Get by In Maths

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Contents

Nomenclature

Symbols

$\vec{r}$	position vector
$\vec{v}$	velocity vector
A	area
A	solar altitude (max.)
a	acceleration
F	force
R_{earth}	Earth radius
s	distance (general definition)
SD	Distance travelled in one day
T	Upper integration limit (regarding time)
T	tension (in rope
t	Time
V	Upper integration limit (regarding velocity)

v	velocity
S	Distance travelled from t=0 to t = T
x	Distance travelled

Greek Symbols

β	compass bearing
δt	small increment of time
δ	indicates small change
δ	solar declination
λ	latitude
ϕ	longitude
τ	torque
θ	vector angle
$\vec{\Delta r}$	Displacement vector
m	indicates optimum

1. Week One: Introduction

A warning. These notes lack what the purists would call "rigour". We have taken lots of shortcuts, and focus on problem solving rather than where things come from (derivation).

We reference appropriate web sites for these details. Nor do the notes cover all Year 1 requirements - this Summer School occupies only a few hours of activity and cannot compete with the 200 hours that will be devoted to the topic of "Engineering Mathematics" in your Year 1, but may prompt you to look further at some of the topics that interest you or challenge you.

Students come from diverse educational backgrounds. If some of the Mathematics we use is familiar to you then great: please reflect on how it can be used to assess real-world things. If you haven't studied it before, or are rusty on parts, please have a look at the resources we refer to and work through some of the examples both now and in your tutorial sessions in October.

We have graded difficulties as follows

- ♣ Basic - ensure you know this material well.

- ♣ ♣ Moderate - this in the OCR and AQA single Mathematics A level. Ensure you know this material well.

- ♣ ♣ ♣ Advanced - this material appears in the Further Mathematics A-level. Please try to understand it in readiness for use in Engineering Science, but don't worry too much (it will be taught in lectures).

- ♣ ♣ ♣ ♣ Please attempt this if you enjoy a challenge (it will appear in Fluid Mechanics near Easter).

1.1. Speaker

The speaker talks about the role of Mathematics in their engineering work.

1.2. The course problem and challenge ♣

We seek to:

- demonstrate how maths is used in engineering

- show you some of the ways Mathematics is used by University of Bristol (UOB) engineering undergraduates, graduates in industry, and researchers

- remind you of some key concepts that will be helpful when you arrive at UOB

- re-familiarise you with common mathematical operations (e.g. BIDMAS and trigonometry) and methods of communicating calculations

We shall introduce several worked examples and problems to illustrate our ideas.

To pull things together, we shall use an 'imagined' version of a sea voyage. This will bring together ideas of trigonometry, calculus, vector manipulation, optimisation, data handling and presentation.

> **Example: A reenactment ♣**
>
> Consider a reenactment of a 19th century sea voyage from Tahiti to Gisborne, New Zealand [a]. There was no GPS on the original voyage. List ways in which mathematics could assist with (1) design of the ship and (2) navigation.
>
> ---
> [a] https://en.wikipedia.org/wiki/First_voyage_of_James_Cook

My answers are as follows. (1) Wind acts on the sails and provides propulsion. We need to resolve forces - forces are vectors. (2) The hull should be sufficiently buoyant to carry the weight of cargo and infrastructure. Estimation is needed. (3) The hull should be strong enough to withstand the above weight and water forces (hydrostatic pressure). (4) Navigation required sightings of landmarks, stars and planets.

Trigonometry (and geometry) come into play. This is true with modern GPS. (5) The trajectory of the ship is in two directions - again vector operations come into play. (6) You need to record, process and report data.

1.3. Estimation ♣

A lot of engineering is about estimating things. We rarely have a precise answer to a problem - there is always some assumption. We often use a very rough estimate as a way of "sanity checking" a more detailed calculation.

For centuries, good engineers have been familiar with basic slide rule functions [1], now captured more conveniently and precisely on a pocket calculator.

To illustrate the layout of a calculation, we shall consider the amount of paint required to paint a room.

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[1] `https://www.sliderules.org/`

Measure a room and estimate the cost of painting the walls magnolia. (My domestic line manager has requested that one wall is to be red.) Warning The following calculations are inaccurate and serve only to illustrate layout.

A model answer should include

- A problem statement. It is a summary showing the essential statements in the question and helps you to check that you are answering the right question, especially in long or complicated problems. In Year 3 the problem statement will be a crucial part of your dissertation, setting the scene so that the audience understands what you are trying to do. *Trivial example: how much will it cost to paint three walls in a room magnolia?*

- A set of drawings, showing what is happening. *Trivial example: the room's floor-plan is sketched below.*
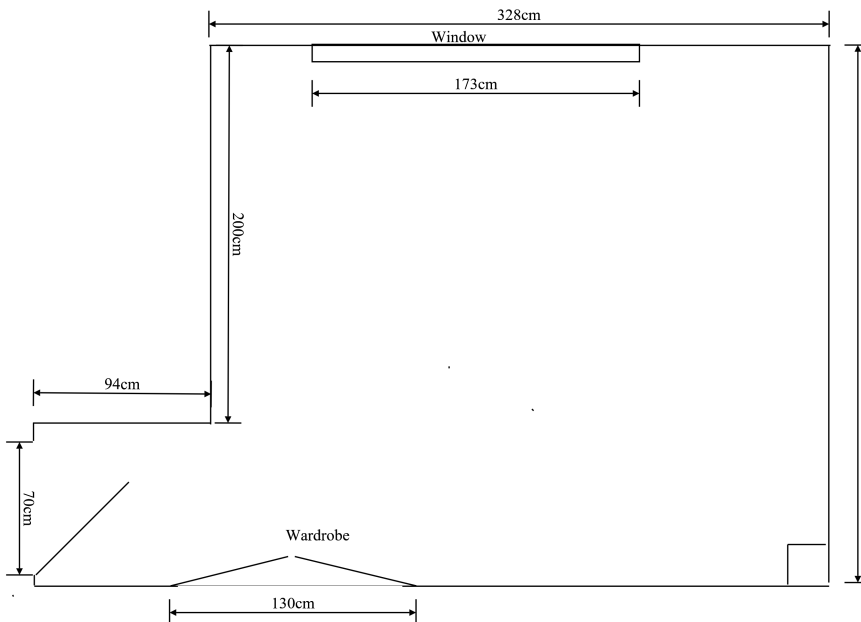

Figure 1: Floorplan of our room (height = 2.31 m)

- A list of assumptions. *Trivial example: We assume two coats of good quality paint. The floor-plan is approximated as a simple rectangle. We don't account for omitted parts (radiators, wardrobe doors, the back wall which is to be painted pepper red), or additional parts (the alcove, various recesses). The costs of cleaning materials and brushes are ignored, and a big assumption is that labour is free.*

- A list of <u>physical laws</u> – e.g. in Engineering one may well use Boyle's Law (It's hard to think of a physical law for painting!)

- A list of <u>physical properties</u> - e.g. in Engineering Young's Modulus of aluminium. *Trivial example: According to Dulux, a small (2.5) litre tin of paint should cover 35 square metres. I should also add a metric for how long it takes me to prepare and paint a certain surface area.*

- A set of <u>calculations</u> – you should explain your thinking so that someone reading your work could check you hadn't made any errors. *Trivial Example: Allowing for two coats, the area to be painted is approximately* $2(coats) \times 2.31m(height) \times 2 \times (3.28m(Length) + 2.84m(width)) \approx 56.5m^2$. *This is 2 tins.* (And never, ever, in an assessment, project report or industrial report write 1.61568 tins!)

- <u>Discussion</u> –how realistic is your answer? It would be worth, for example, confirming an "optimum cruising speed" against previous experience, or a second (or third) calculation using different principles. Consider a sanity check. *Trivial Example: I did carry out a more detailed calculation ($43m^2$ coverage $\implies$ two tins. I also compared the number of tins with those required in other rooms. I might have added an extra 20% to allow for safety (still two tins).*

1.4. Examples of estimation ♣ ♣

Students who work through these calculations are invited to participate in the course blog.

> **Example: Weight control and mechanical work ♣ ♣**
>
> Under my current regime of exercise and calorie intake I have kept to my target body weight of 10 stone. I have an uncontrollable desire to eat one additional Tesco Garibaldi biscuit each day. To retain the same body weight, how many additional press-ups will I need to do? (Assume that my metabolism acts with an efficiency of 19% to convert food energy to mechanical energy. Assume that each press-up raises my centre of gravity by 25 centimetres.)

Problem statement: Find the number of extra press-ups to dissipate a the calorific energy in one biscuit.

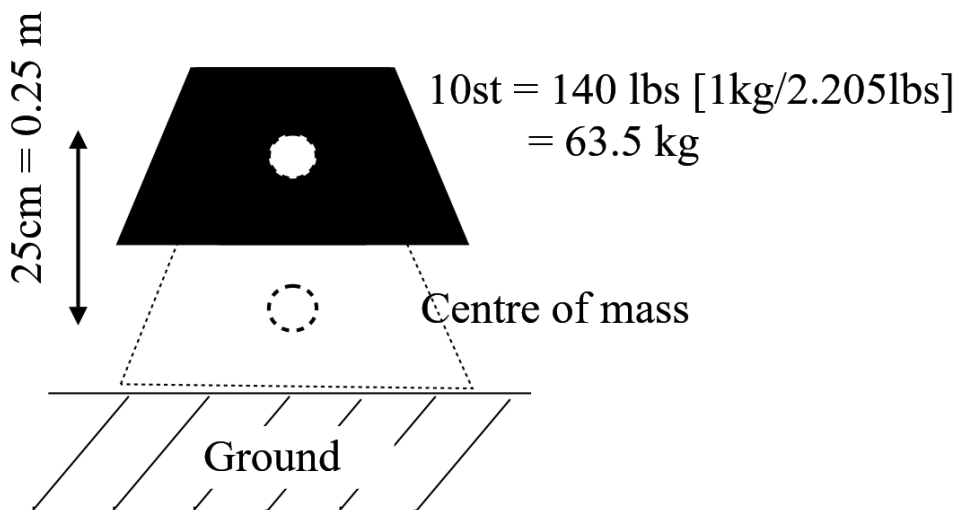Diagram: Simplify the problem as lifting a mass N times.

Figure 2: Work done in press-ups

Assumptions: Assume all mechanical work concerns raising mass - ignore friction work and any kinetic energy.

Physical Laws: Gravitational energy of a mass, $GE = mgz$.

Properties: Gravitational acceleration, $g = 9.81ms^{-2}$. Chemical energy in a biscuit, $CE = 41kcal * 4.2kJ/kcal = 172kJ$ [2]/

Calculation: The mechanical energy provided by one biscuit, $ME_{biscuit}$

$ME_{biscuit} = 0.19 \times 172 = 33kJ$

The mechanical energy for one press up is equal to the change in gravitational potential energy

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[2]https://www.tesco.com/groceries/en-GB/products/254921673

$ME_{press} = mgz = 63.5 \times 9.81 \times 0.25 = 156J$

The required number of press-ups is.

$N = \frac{ME_{biscuit}}{ME_{press}} = \frac{33kJ}{156J} \times \left[\frac{1000J}{1kj}\right] = 212$

Discussion:A key point is that I have to research the calorific value of a Garibaldi biscuit (be very careful with units, because many nutrition websites confuse calories and kilocalories). I also have to convert stones to kg - unit conversion is an important topic covered in Dr Lancastle's sessions on Physics.

Problem statement: Find the capacitance of a supercapacitor (in Farads) (from which charge $Q = CV$ and energy $Q = 0.5CV^2$)
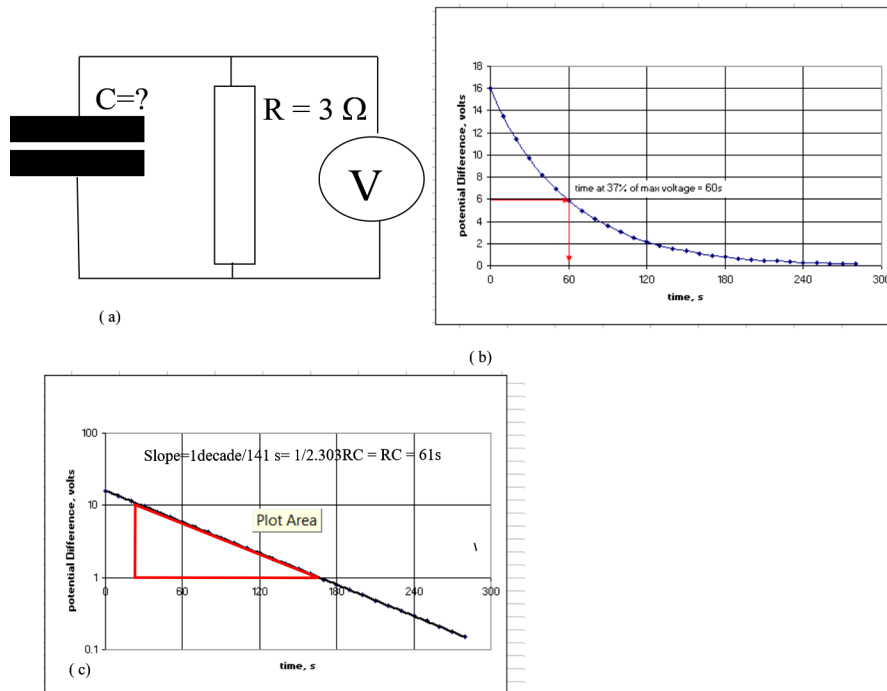

Figure 3: Discharge of a capacitor (a) electrical circuit (b) 'recorded' plot of voltage decay, initially voltage is $V(0) = 16 volts$. At V=0.372*16 volts read $\tau = 60s$ (c) same data on a log-linear plot. Slope = 1 decade/138seconds, $\tau = 138/2.303 = 61s$.

Assumption: The electrical resistance is constant (notwithstanding comparatively high currents and heating).

Physical laws: The voltage across the capacitor decays exponentially according to:

$$V = V(0)exp(-\frac{t}{RC}) = V(0)exp(-\frac{t}{\tau})$$

where $\tau = RC$ is a time constant

Calculation When one time constant has passed we find $t = \tau = RC$ and $V = 0.372V(0)$ (that is, $0.372 = e^{-1}$ where e represents Euler's number). So at $t = 60s$ $V = 0.372 \times 16 volts = 5.9 volts$ produces $t = \tau = 60s$ and C = 60s/3ohm = 20 Farads. A device with such a huge capacitance is termed a supercapacitor.

Figure (c) shows a different approach. Here we use a logarithmic y-axis (log to base 10). Let increments

in the y-axis be counted in decades, 0.1 to 1.0 is one decade, 1.0 to 10.0 is one decade etc. Then $slope = number\ of\ decades/\Delta t = 1/138$. Now Excel (or printed log-linear paper) uses base 10; $log_e(10) = 2.303$. So we get $\tau = 1/(2.303 \times slope) = 138/2.303 = 61s$

Problem statement: Distance travelled along arcs formed by lines of latitude and longitude.

Diagram:
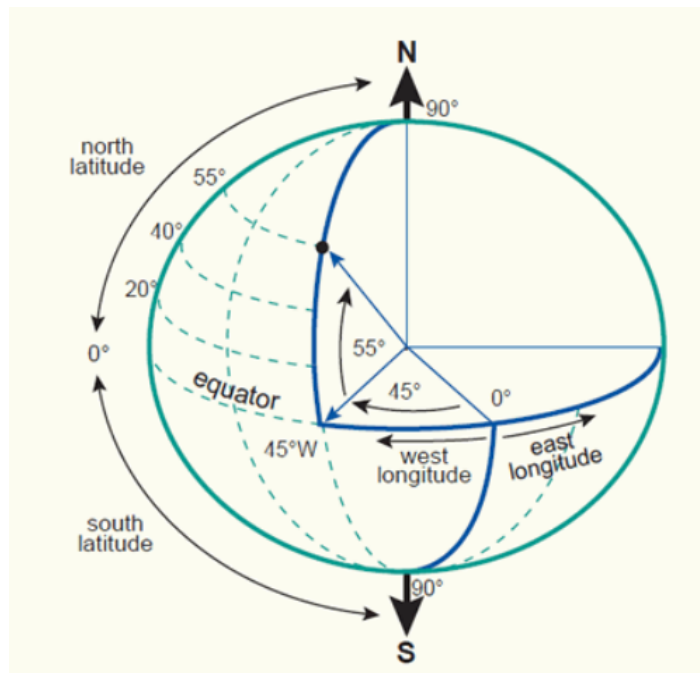
Figure 4: Point at longitude −45° and latitude 55°

Assumptions: Treat the earth as spherical.

Principles: The length of an arc is $s = r\theta$, where the angle θ is measured in radians. Basic trigonometry [3]

Properties: The earth's radius is approximately 3440 nautical miles (6371 km). $1 rad = 180/\pi = 57.3°$

Calculation: Moving north the centre of curvature is that of the earth, and the radius is $3440 N.miles$. The distance travelled is,

$$s = R_{earth}\theta = 3440 \times \frac{3}{57.3} = 180\ N.miles$$

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[3] https://www.bbc.co.uk/bitesize/guides/zq8yg82/revision/1

Take care with east/ west movement, because a rather different radius is needed. The centre is point B on the sketch below , PB being perpendicular to the earth's North-South axis.
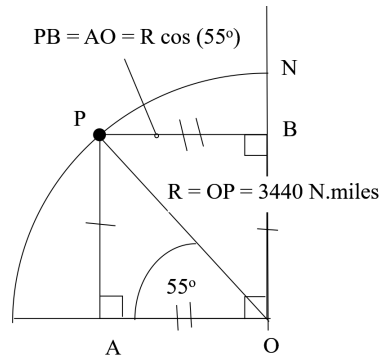
Figure 5: Cross section through the earth, showing a smaller radius PB for lines of latitude.

$$s = distance\, PB \times \theta = 3440 cos(55°) \times \frac{3}{57.3} = 103\ N.miles$$

Discussion: The Nautical mile was conceived to correspond to a one second arc along a line of longitude. Hence one degree corresponds to 60 nautical miles, confirming our estimate of $3 \times 60 = 180$ nautical miles in the northerly direction.

1.5. Homework ♣ ♣

This week concerns preparatory work for a theme on a re-enactment of a sea voyage from Tahiti to Gisborne in New Zealand. We shall go on to explore trigonometry, units conversion, vectors, differential quantities, use of computer packages (Excel or similar). These are related to engineering drawings, surveying, navigation (self driving vehicles), robotics, dynamics, and Fluid Mechanics.

Homework: New locations ♣ ♣

(A) Complete the multiple choice questions on the next page. (B) Visit `https://demonstrations.wolfram.com/DeterminingLatitudeFromMaximumSolarAltitude/`. Try out locations below the Tropic of Capricorn and explain the relationship between angles of declination, maximum solar altitude and latitude (C) For the first few days of the voyage maximum solar altitudes (at mid-day) are tabulated below. Calculate the missing entries in the table. (D) In the workbook MathHomework.xls (see the first spreadsheet) edit and correct the formula for latitude in column E (labelled "wrong").

Table 1: Sextant measurements of solar altitude

Day	Month	Day Number	Altitude, °	Declination °	Latitude °	
14	8	226	51.8	14.2	-24.0	
15	8	227	51.1	13.9	?	
16	8	228	50.3	13.5	?	
17	8	229	50.0	13.2	?	
18	8	230	49.5	?	?	
...	...	...	...	...	...	...

<u>Tips and hints</u>

For part B, when working south of the Tropic of Capricorn select the north-direction button. The two sliders are very sensitive.

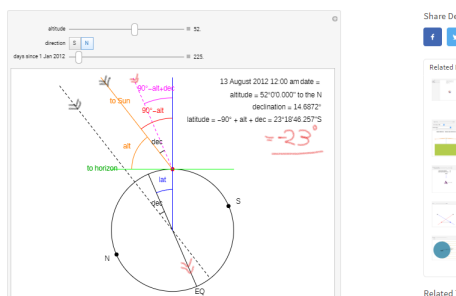Figure 6: Screen capture from Wolfram solar demonstration. Make sure you select the North direction for Southern hemisphere. Chevrons indicate parallel lines. Note that the dashed lines are construction lines, parallel to the sun's rays (black) or a line passing through the equatorial plane (purple)).

For part (C) We have calculated nearly all declination angles, δ, for you. (We leave one blank space so

you can practice trigonometry functions.) It is subtended between the suns rays and a line passing through the equatorial plane. [4] [5] . The declination angle follows

$$\delta = -23.45 cos(\frac{360°(d+10)}{365})$$

where d is the day number, d=1 on New Year's day. The latitude, λ, follows from

$$\lambda = A + \delta - 90°$$

where A is the altitude. (Notice how the latitude can be expressed as $-24°$ or $24°\,S$)

For part (D) tips on using spreadsheets are at `https://exceljet.net/excel-formulas-and-functions` and `https://edu.gcfglobal.org/en/excelformulas/relative-and-absolute-cell-references/1/`. If you do not have Microsoft Office installed on your computer then there are free options, see for example `https://www.techrepublic.com/article/5-free-alternatives-to-microsoft-excel/`. I use Google Sheets, Libre Office and WPS; these should be OK for this Summer School.

In MathHomework.xls you will see that the wrong formula for latitude in cell E4 is

=180+C4-B4

You should correct this and copy the correction into other cells using a *relative* copy.

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[4]`https://susdesign.com/popups/sunangle/declination.php`
[5]Scroll down to the animation at `https://www.pveducation.org/pvcdrom/properties-of-sunlight/declination-angle`

Record in your log book the responses to the following. We shall discuss responses next week.

I wish to know the height of a tree, but cannot climb it. I stand 10 metres from the tree's base and measure the angle of elevation of an virtual line from my toes to the top of the tree. The angle is 60°. Is the height.

- 10 m

- $10 \times tan(60°)m$

- $10 \times sin(60°)m$

- $\frac{10}{sin(60°)}m$

A can of soup is $80°C$ hotter than its surroundings. It is left to cool, and after five minutes is $29.76°C$ hotter than its surroundings. After a further five minutes it is $11.07°C$ hotter than its surrounding. Yet another five minutes elapses - what then is the temperature difference?

- $80 \times 0.372 \times 0.372 \times 0.372°C$

- $10 \times 0.372 \times 0.372°C$

- $0°C$

- none of these

Which of the following statements is correct concerning the previous problem?

- The temperature difference decays exponentially, following Newton's law of cooling.

- We use a quadratic formula to get the temperature difference.

- Every five minutes, we divide the temperature difference by Euler's number

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With regards to Figure 7 which of the following statments are true?

a $\beta + \theta = \pi/2$

b $sin(\beta) = cos\theta$ and $cos(\beta) = sin(\theta)$

c for a distance s from the origin and along the red arrow, $sin(\theta) = y/s$
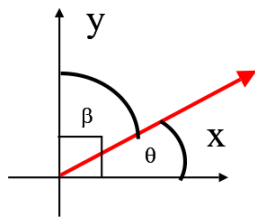
d all the above

Figure 7: Sketch for multiple choice question.

2. Week Two: Movement,calculus and vectors

We start with a summary of homework responses, and in particular discussion of assumptions made and comparisons of dead-reckoning latitudes with recorded values.

We note some things about the homework so far:

- Displacement and velocities (and later forces) are in two or more directions. This introduces the concept of vectors.

- Things like speed change with respect to time. To get displacement we have to sum large number of small contributions, assuming that over a small time interval the changes in certain variables (e.g. latitude) can be ignored. This leads us to integration and further to calculus in general.

2.1. Speaker

The speaker talks about the role of Mathematics in their engineering work.

2.2. Laws of motion ♣ ♣

In engineering motion applies to road vehicles, aircraft, robots, engines, and earthquakes. In this part we consider distance, speed and acceleration - direction (ship's bearing) will be ignored for the time being. Distance, speed and acceleration are related to each other by differentiation and integration. Consider an extract from the ship's log, Figure 8. The circled items are the hour (H), speed in knots (K), and course. One knot is equivalent to one nautical mile per hour. The course changes somewhat but for the purposes of this discussion let us take it as constant for the day, say north-north-west (a bearing of $332°$). At the bottom of the page we see, in pencil, the number 55. I suspect that this is equal to the daily distance in nautical miles. (Found by counting the total number of knots in the line that was paid out in 24 hours. The recorded speed is imprecise - "2 knots" might be anywhere between 1.5 and 2.5 knots).

The navigator could add up the hourly measurement of speed to get the distance travelled (this is not distance in a straight line, but at this stage let us not worry excessively about direction). The operation is *summation*. This distance travelled in one day is,

$$SD = \sum_{1}^{24} v_i \delta t = 2knots \times 1hour + 2knots \times 1hour + 2knots \times 1hour + 1knot \times 1hour + ...2knots \times 1hour$$

where v is velocity and δt is the one-hour-long time interval. If we see these the time interval as infinitesimally (=very, very) small then we have a special form of summation called "integration". Of course, with modern electronic meters we could easily record many times per second, and for practical purposes the time increments are infinitesimal.

Figure 8: Extract of ship's log, 2 September. Circled items are time (H), speed in knots (K), course and distance travelled in day. The latitude at mid-day was $39°05'$
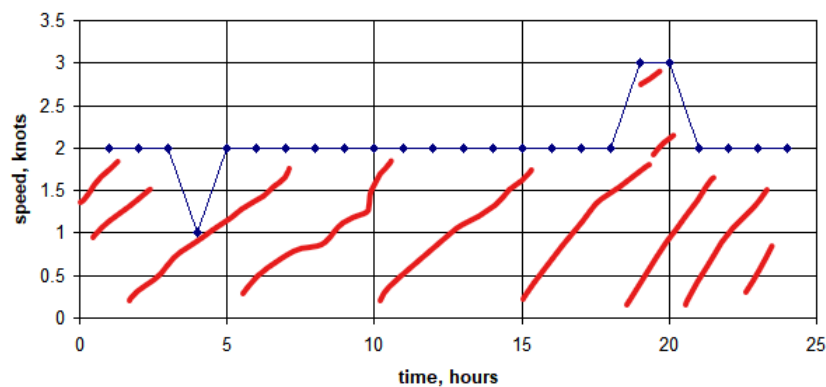
Figure 9: Plot of recorded velocities. Distance travelled = area under curve.

$$SD = \int_0^T v\,dt$$

with $T = 24\,hrs$.

Integration yields the area under the curve. E.g. distance is the area under the distance time curve below. (Or, in this case, piece-wise continuous lines).

Integration replaces multiplication. We often want to say "work = force multiplied by distance" or "distance = velocity multiplied by time". But force and velocity frequently tend to change and we must

replace "multiplied by" with "integrated with respect to".

We refer to acceleration as approximately the change in velocity per hour

$$a \approx \frac{\delta v}{\delta t}$$

Now admittedly the time for the ship to accelerate from say two knots to three knots is probably only a few minutes - far less than the time interval between speed measurements, $\delta t =$ one hour But strictly for the purposes of illustration, however, we shall make use of the available data. If we were then to use recorded data apparent accelerations would be 0, -1 or +1 nautical miles per hour per hour. There are only four lines on Figure 9 where slope and hence acceleration are non-zero.) If we could measure velocties every second (say) then we could legitimately write the instantaneous acceleration as,

$$a = \frac{dv}{dt}$$

Differentiation is the slope of a curve. It replaces division. In the above, rather than saying "acceleration = velocity divided by time" we differentiate because different accelerations apply at different times.

The SUVAT equations give nice examples of integration, differentiation and also separation of variables [6]. Let us start with the definition of acceleration, forming the simplest example of an ordinary differential equation.

$$a = \frac{dv}{dt}$$

There is a technique known as *separation of variables*. Separate variables so that terms with v are on the left hand side and terms with t are on the right hand side. Multiply throughout by *dt* and divide throughout by *a*

$$dv = adt$$

Integrate this between limits. I'll use capital letters to show limits - the start and end conditions - so $v = U$ at $t = 0$ and $v = V$ at $t = T$.

$$\int_U^V dv = \int_0^T adt$$

Notice that $\int dv = \int 1dv = v + const$. Provided that acceleration a is a constant we solve and manipulate to get,

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[6]See, for example `https://i-want-to-study-engineering.org/q/derive_suvat/`.

$$V - U = a(T - 0) \implies V = U + aT$$

This is true for all times t in the range from 0 to T (or $0 \leq t \leq T$), so one can also write,

$$v = U + at = \frac{dx}{dt}$$

Then with further separation of variables, integrations and manipulation,

$$S = uT + \frac{1}{2}aT^2$$

Problem statement: Estimate velocity and mean velocity 12.5 s after acceleration from rest: acceleration is a function of velocity.

Diagram:

a=2-0.08v

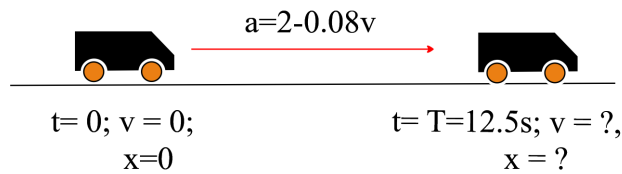t= 0; v = 0;   t= T=12.5s; v = ?,
x=0   x = ?

Figure 10: Acceleration of a vehicle

Assumptions: None because acceleration is prescribed.

Physical laws: No physical laws because acceleration is prescribed. Mathematical methods are separation of variables and integration.

Calculation: Start with

$$\frac{dv}{dt} = 2 - 0.08v$$

Separate variables - velocity on LHS and time on RHS. Then integrate, with velocity v=V at time t=T.

$$T = \int_0^T dt = \int_0^V \frac{1}{2 - 0.08v} dv = \int_0^V \frac{1}{a} dv$$

The graphical way: For the simple function above this seems rather time consuming - the usefulness becomes evident with complicated functions - for example in an earlier version of these notes we tried $dv/dt = 2 - 0.0032v^2$, which offers a better representation of drag forces. If you wish to investigate further our working is in the online file of examples *mathsSummerSchoolExamples.xls*. Consider the plot of reciprocal acceleration, $1/a = \frac{1}{2-0.08v}$, versus velocity. For any velocity, v, Figure 11 part(a), the area under the curve gives elapsed time, t. This area can be found by the trapezium rule or roughly by counting rectangles. On Figure 11 (a) $1\,rect = 2 \times 0.5 = 1s$, so the time to move from rest to 2m/s is 1.03 s (a little more than one rectangle), the time to move from rest to 4m/s is $1.03 + 1.12 = 2.15s$ etc. With this information we get the velocity-versus-time curve on part (b) - at time $t = 12.5s$ the corresponding velocity is $\underline{V = 15.8m/s}$. Each rectangle now corresponds to a distance of $25m$ and the area under the curve gives the distance travelled after $T = 12.5s$, $\underline{S = 115m}$. The average velocity is represented by the rectangle with green dotted lines,

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the area of which is the same as the (shaded) area of integration.

$$v_{avg} = \frac{S}{T} = \frac{115m}{12.5s} = 9.2m/s$$

(a)

(b)
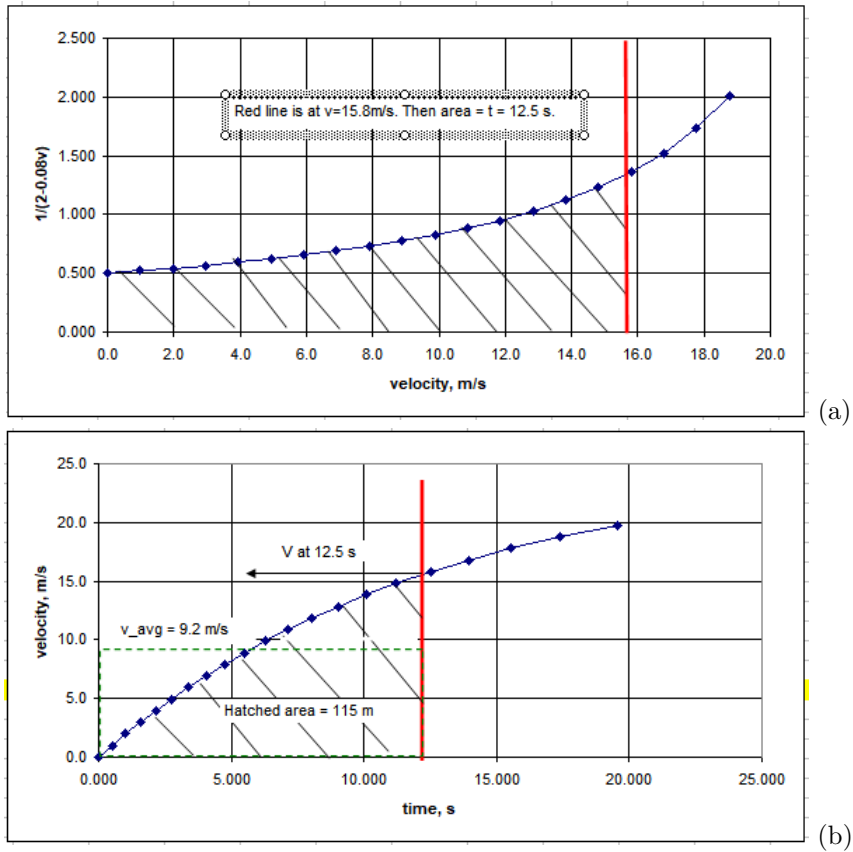
Figure 11: Integration giving (a) velocity at area under curve = t = 12.5s (b) instantaneous and average velocity at t = 12.5s

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The hard way: Some tasks (although not this one) feature some unpleasant integration, but the good news is that there are many computer tools to do this for us. We'll use Wolfram Alpha [7] - see Figure 12 - but you will also come across MAPLE and MATLAB.

$$T = \int_0^V \frac{dv}{2 - 0.08v}$$

Figure 12: Integration of $1/(2 - 0.08v)$.

A bit of further arithmetic yields,

$$T = -12.5 log_e(1 - 0.04V)$$

And more generally

$$t = -12.5 log_e (1 - 0.04v)$$

This is *implicit* - time is a function of velocity but we need the inverse function with velocity as the subject.

$$v = 25 - 25\,e^{-t/12.5}$$

(Or, try typing this in Wolfram Alpha - solve t=-12.5*log(1-0.04*v) for v)

We can find the distance travelled by integrating velocity to obtain,

$$S = 25 \int_0^T (1 - exp(-t/12.5))dt = 25(T + 12.5e^{-T/12.5} - 12.5)$$

If $T = 12.5s$:

- $V = 25 - 25e^{-1} = \underline{15.8s}$ (= graphical solution)

- $S = 25(12.5 + 12.5e^{-1} - 12.5 = \underline{115m}$ (=graphical solution)

Discussion: It is reassuring that our two answers agree to three significant figures.

2.3. Dimensions in space ♣ ♣

There are several vector calculators online. We shall use `https://onlinemschool.com/math/assistance/vector/`. Another useful reference is `https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/4-1-displacement-and-velocity-vectors/`

For purposes of discussion locations on a map offers a useful illustration of *position vectors*. Provided that distances are small the earth's curvature may be ignored so that latitude and longitude are easily related to x- and y- co-ordinates. For example, if the origin is taken as the bottom left hand corner of a map x- and y- co-ordinates define any point on the map. For the general vector notation shown on Figure 13

- $\vec{r}$ denotes the position vector relating the blue dot to origin O;

- it is *resolved* into two (or three) components, denoted as r_x and r_y;

- the vector has a *magnitude*, shown as either r or $|\vec{r}|$;

- it has a *direction*, shown as an angle θ or $\angle \vec{r}$, measured *anticlockwise* from the x-axis;

- the vector can be written in terms of *basis vectors*, $\vec{i}$ and $\vec{j}$.

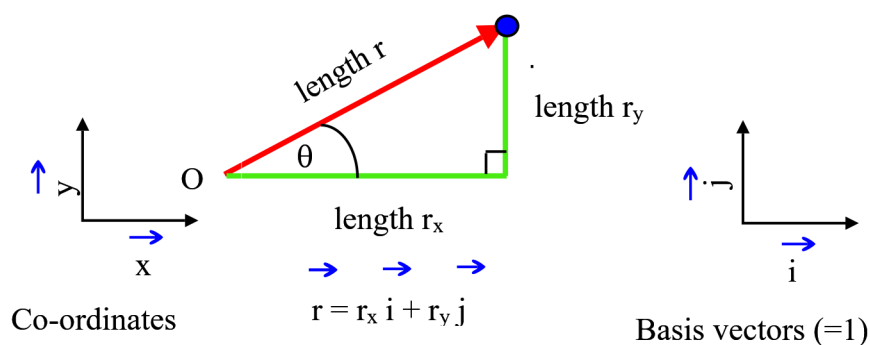Figure 13: Vector shown as multiple of basis vectors, so $\vec{r} = r_x \vec{i} + r_y \vec{j}$

For the special instance of the position vector the components are frequently termed x, y. Differently to a vector angle, a compass bearing, β, is measured from North (y-axis) in a clockwise direction. The vector magnitude and angle follow from,

$$r = \sqrt{r_x^2 + r_y^2}$$
$$\theta = tan^{-1}(\frac{r_y}{r_x})$$

If, for example, the position (on a map) is measured in kilometres then the basis vectors are employed to describe the vector thus,

$$\vec{r} = r_x \vec{i} + r_y \vec{j} \ km$$

The basis vectors have a magnitude of one, $i = j = 1$, and have no units. This means that for different types of vector - position, velocity, force - we can use the same basis vectors (as written above). The units are written at the end of the right hand side of the equation.

Frequently trigonometric functions are used to write vectors in the form

$$\vec{r} = r \cos\theta \ \vec{i} + r \sin\theta \ \vec{j}$$

The *displacement vector* is the difference between the position vectors at two points.

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

The magnitude could be written as Δr but is also known as distance, s

We shall use compass bearing, β, in the homework. This is measured clockwise from north. It is useful to note $\cos\theta = \sin\beta$ and $\sin\theta = \cos\beta$ (see the examples and homework for Session 1). So for navigation purposes a special definition is

$$\vec{r} = r\sin\beta \ \vec{i} + r\cos\beta \ \vec{j}$$

We shall define our own special symbol, SD, for the distance travelled in 24 hours.

<div style="border: 1px solid; background: #fffbe6;">

Example: Velocity vectors and basis vectors ♣ ♣

Consider the ship's log shown as Figure 8. At 02:00 hours and 17:00 the log shows a bearing (β), measured clockwise from the northerly direction, and (in effect) a vector magnitude (or speed). Show the velocities using basis vectors pointing in an easterly direction ($\vec{i}$) and in a northerly direction ($\vec{j}$).

</div>

Both recorded speeds (or velocity magnitudes) are 2 knots. At 02:00 hours the bearing is NNW or $\beta = 332$ ° (clockwise, from North) [8]. At 17:00 hours the bearing is NWBN or $\beta = 298°$. Figure 14 shows the 02:00 hours bearing. (I also show geometric manipulation to get the vector angle of 118°.)
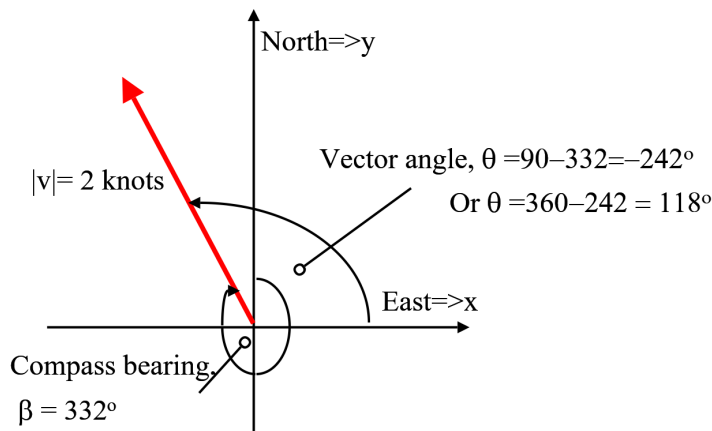
Figure 14: Ship bearing (refers to 02:00 hrs in ship's journal)

So at 2.00 hours

$$\vec{v} = 2sin(332°)\vec{i} + 2cos(332°)\vec{j}$$

$$\vec{v} = -0.939\vec{i} + 1.766\vec{j}\ knots$$

(A useful sanity check is retrieving the original speed, so $\sqrt{0.939^2 + 1.766^2} = 2$.)

At 17:00 hours: $\vec{v} = -1.766\vec{i} + 0.939\vec{j}\ knots$

<div style="border: 1px solid; background: #f2f2f2;">

Example: Vector addition to give resultant vector♣ ♣

The ship rescues a castaway from a desert island. The castaway reports that, from the island, they have frequently observed driftwood floating in a north-east direction at 0.8 knots. Given that the ship's navigator has used a Dutchman's log to carry out dead reckoning, re-estimate the above velocities relative to the island.

</div>

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[8]https://en.wikipedia.org/wiki/Points_of_the_compass

Problem statement: Find true ship's velocity in the absolute frame of reference.

Diagram and principles: The problem is one of frame of reference - Figure 15. The shipboard observer "sees" the vessel velocity *relative* to the ocean (red arrow) whereas the onshore observer sees the velocity of the ocean (purple arrow) and hence the velocity of the frame inside which the ship finds itself. To get the true velocity of the ship one must add the drift velocity (purple arrow) to the apparent velocity (red arrow). This is vector addition - two effects, each in itself relating to two directions, are added to give the "absolute velocity"(orange arrow). The vector addition yields the *resultant vector*.
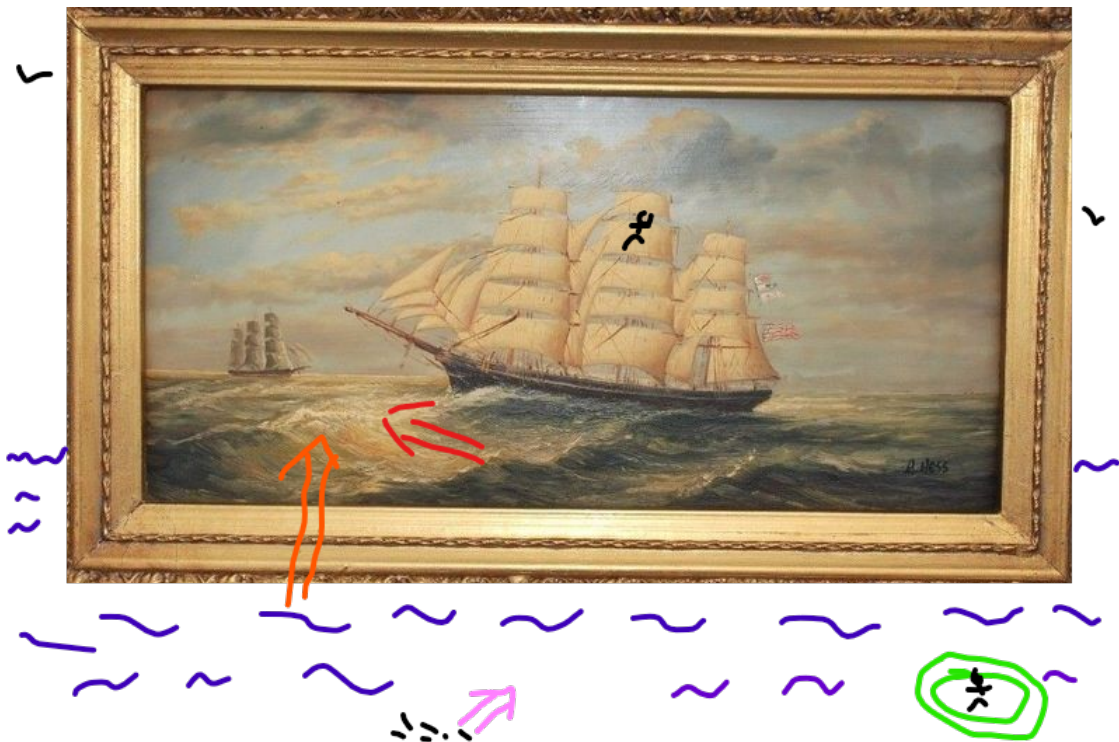
Figure 15: Frame of reference. The observer on the ship experiences a velocity of v_{app}, the red arrow, relative to the surrounding ocean and as evidenced by the compass and Dutch log. The observer on land sees that the ocean itself moves with velocity v_{drift}, the purple arrow, relative to land and as evidenced by the movement of flotsam. The true movement of the ship, relative to land, is the sum of these two and thus the orange arrow.

Assumptions: The ocean drift velocity is uniform for the ship course under consideration. The driftwood moves at the same speed as ocean currents.

Principles: Vector addition.

Calculation: The vector addition is shown more formally in Figure (16). (One aligns east with x and north with y.) The indicated vector angles are found from the compass bearing and speed (="velocity magnitude"). At 02.00 the apparent velocity has already been calculated as

$$\vec{v}_{app} = -0.939\vec{i} + 1.766\vec{j} \ knots$$

The velocity of the ocean currents is treated as equal to the observed velocity of driftwood and is

$$\vec{v}_{drift} = 0.8 \times sin(45°)\vec{i} + 0.8 \times cos(45°)\vec{j} \ knots$$

$$\vec{v}_{drift} = 0.566\vec{i} + 0.566\vec{j} \ knots$$

The resultant velocity, in the absolute frame (= "island frame") of reference follows.

$$\vec{v}_{abs} = \vec{v}_{app} + \vec{v}_{drift} = -0.373\vec{i} + 2.332\vec{j} knots$$
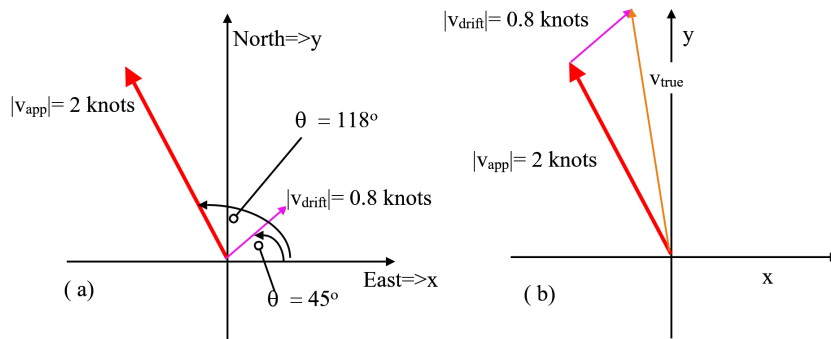
Figure 16: Vector addition (a)two velocities with same origin, red is measured ships velocity and purple is ocean current velocity (b) velocities are arranged head to tail to indicate the (orange) vector addition - the true ship's velocity

From Pythagoras, the speed/ velocity magnitude is $v_{abs} = \sqrt{0.373^2 + 2.332^2} = 2.36m/s$. The new bearing is $\beta = tan^{-1}(\frac{-0.373}{2.332}) = -9° = 351°)$. Note also: (1) the arctangent is also equal, misleadingly, to $171°$ [9]; (2) whereas Figure 16 shows the head-to-tail method there is also the parallelogram method [10].)

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[9] https://www.bbc.co.uk/bitesize/guides/ztj6y4j/revision/6
[10] https://www.ae.msstate.edu/vlsm/forcesys/vector_review/parallel.htm

2.4. Examples ♣ ♣

A sailboard moves at $20km/hr$, the vector angle is $60°$ taken anticlockwise from the easterly direction. Observed from the shore, the wind speed is $30km/hr$ in a direction $160°$ (taken anticlockwise from the easterly direction). What is the "apparent wind"[a], i.e. the wind velocity perceived by the windsurfer?

[a]https://en.wikipedia.org/wiki/Apparent_wind

Problem statement: Find wind velocity observed by windsurfer. Diagram:

(a) View from shore   (b) Viewed from boat
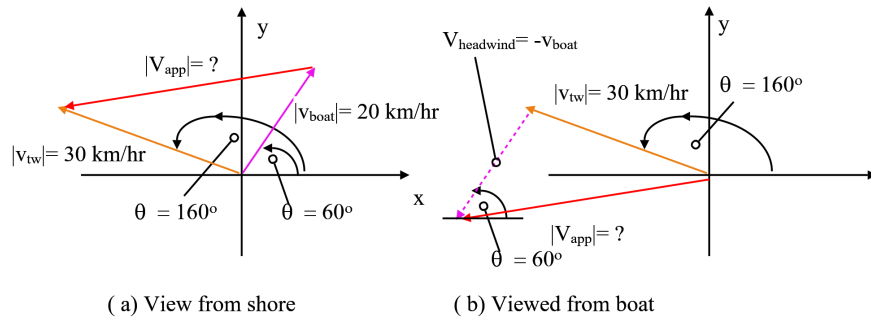
Figure 17: Apparent wind in two frames of reference (a) true wind and boat velocities are percieved from the shore (the apparent wind is the difference between the two) (b) headwind and apparent wind velocities are perceived from the boat (apparent = true + headwind

Assumptions: Discussion is restricted to boat and wind vectors.

Principles: Vector addition/ subtraction. We use vector angle (rotating anticlockwise from x-axis) rather than bearing.

Calculation: Describe boat and true wind velocities in terms of basis vectors - $\vec{v} = v(cos(\theta)\vec{j} + sin(\theta)\vec{j})$.

$$v_{boat} = 10.0\vec{i} + 17.3\vec{j} \ km/hr$$

$$v_{tw} = -28.2\vec{i} + 10.3\vec{j} \ km/hr$$

In the shores's reference frame (part a on Figure 17), subtract the boat velocity from the true wind

$$v_{app} = v_{tw} - v_{boat} = -38.2\vec{i} - 7.0\vec{j} \ km/hr$$

The velocity magnitude (speed) and vector angle are

$$\angle \vec{v}_{app} = tan^{-1}(-38.2/-7.0) = \ 190° \qquad v_{app} = \ \sqrt{38.2^2 + 7.0^2} = 38.8 \ km/hr$$

Alternatively use the concept of "headwind" , that is $(-1) \times \ boat \ velocity$. (Figure 17 (b)).

<div style="background-color:yellow">Example: Mass supported by two ropes ♣ ♣</div>

A weight of 2000 N is held steady by two ropes, both inclined to the horizontal. The weight is represented as a vector, $\vec{W} = -2000\vec{j}$ N The tension in rope 1 is 800 N, and the rope is inclined at $40°$ to the horizontal. What is the tension in rope 2?

Problem statement: Find tension in one of two supporting ropes.

Diagram:
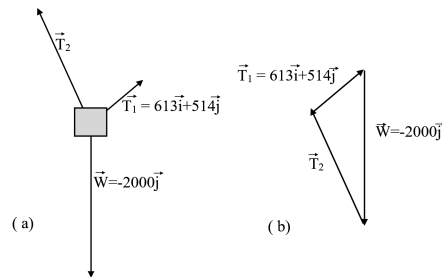
Figure 18: Supporting a weight: (a) as a free body diagram (b) vectors rearranged to indicate zero net force on object

Assumptions: Object of interest is static, so forces are balanced.

Physical laws: Newtons First Law - no movement in x- or y- directions because net force is zero.

Calculation: Known vectors are expressed as follows.

$$\vec{W} = -2000\vec{j}\ N$$
$$\vec{T}_1 = 613\vec{i} + 514\vec{j}\ N$$

The weight is held steady so the net force is zero. This is shown in two forms on Figure 18 : (a) as a free-body diagram [11] (b) the vectors are rearranged head-to-tail in a closed triangle, indicating zero net force. Rearrangement gives the second tension

$$0 = \vec{W} + \vec{T}_1 + \vec{T}_2$$
$$\vec{T}_2 = -\vec{W} - \vec{T}_1$$
$$= +2000\vec{j} - 613\vec{i} - 514\vec{j}$$
$$= -613\vec{i} + 1486\vec{j}$$

Also $|\vec{T}_2| = 1607N$ and $\angle\vec{T}_2 = 112°$.

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[11]https://www.wikihow.com/Draw-a-Free-Body-Diagram

The owner of a new, spring-loaded "bow-classic bullworker" calibrates it by applying a force of 100 Newtons. The handles are then moved 5cm closer to each other than normal. How much work is done to move the handles from this 5cm position to one where they are 30 cm closer than normal?

Problem statement: Work to move a compressed spring from position 1 to position 2.

Diagrams
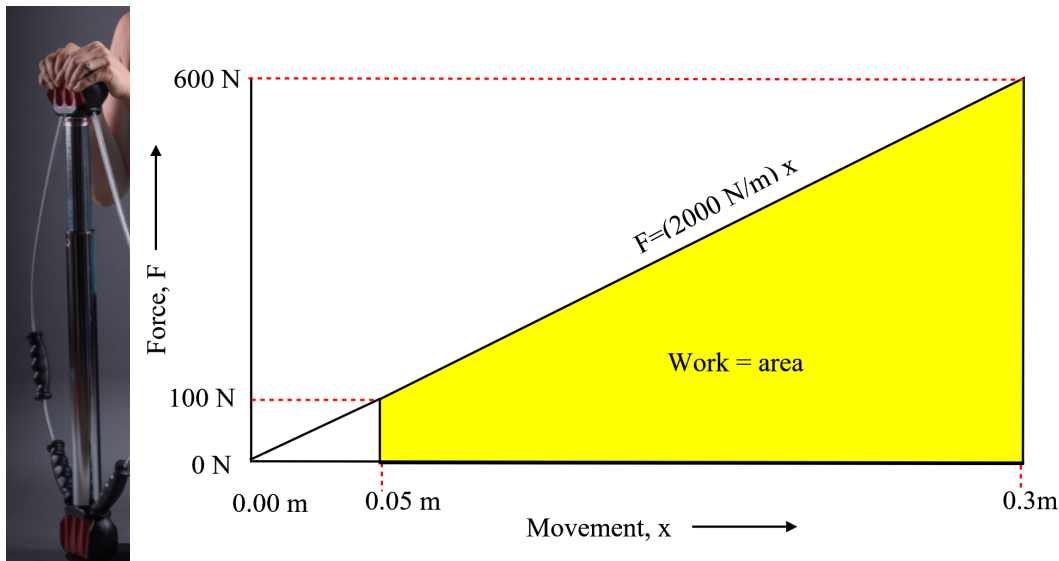
Figure 19: Compression of a spring (a) the bullworker (b) plot of force versus distance. Force is zero when uncompressed and 100 N when x = 5cm. We want the area under the line from 5cm to 30 cm

Assumptions: The spring force is in proportion to its displacement (Hooke's law).

Physical Laws: Hooke's Law - spring force is in proportion to displacement. Work definition - work = constant force times distance or area under force-versus-distance curve.

Calculation: The force-movement line passes through the origin (zero force/ zero movement) and from the one known point one infers $F = 2000(N/m)x$. The work is

$$W = \int_{0.05}^{0.3} F dx = 2000 \int_{0.05}^{0.3} x dx = \frac{2000}{2}[x^2]_{0.05}^{0.3} = 1000 \times (0.3^2 - 0.05^2) = \underline{87.5J}$$

Discussion: The work has a term looking like $2000(x_2^2 - x_1^2)/2$. This is equal to

$$\frac{(2000x_2 + 2000x_1)}{2}(x_2 - x_1) = \frac{F(at\ x_2) + F(at\ x_1)}{2} \times (x_2 - x_1) = average\ force \times distance\ moved$$

.

To make our lives easier, we shall (1) start the story on the Tropic of Capricorn, a day's voyage from the start point (2) assume that every 24 hours - from midnight to midnight - we have available the total distance travelled and the "average" bearing. This distance travelled was measured by a Dutchman's log (or common log), and converted to nautical miles [12]. [13] [14]

Homework: New locations ♣ ♣

Table 2 below contains extracts from the ship's log [a]. Use dead reckoning to find the latitude and longitude at the end of each of the days tabulated. (A) List assumptions and mathematical principles used in your calculations. (B) Complete the missing items in Table 2 (C) The longitude and latitude are calculated in the spreadsheet MathHomework.xls (see tracks A). The longitude is **incorrect**. Edit the cell formula and correct this term. (If item C proves too difficult you may wait to next week.)

[a]`https://nla.gov.au/nla.obj-517532598/view?partId=nla.obj-558521253`

Table 2: Extracts from ship's log.(Also shown in the Excel spreadsheet). A, B are date, C is nautical miles travelled in one day, D is compas bearing, E and F are latitude and longitude from dead reckoning.

Col. A	Col. B	Col. C	Col D	Col. E	Col. F
Day	Month	N. miles	Bearing, °	lat. °	Long. °
start				-23.5	-147.9
14	8	70	169	-24.65	-149.46
15	8	92	288	?	?
16	8	91	248	?	?
17	8	36	214	?	?
18	8	47	135	?	?
19	8	158	135	?	?
...	...	...	...	...	...

Tips and hints

Review the longitude an latitude example on page 12, noting that the arc radius on a line of latitude decreases as one moves northwards or southwards. Figure 20 shows (a) compass bearing, and relationship between distance travelled and its easterly and northerly components (b) the co-ordinate system for longitude (symbol λ) and latitude (symbol ϕ).

We shall complete the first line of the table, 14 August. After passing the Tropic of Capricorn the ship completes 70 nautical miles before midnight, on a bearing of 169°. The distance travelled can be treated as a *displacement vector* that is generally expressed as

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[12]`https://oceanservice.noaa.gov/facts/nauticalmile_knot.html`
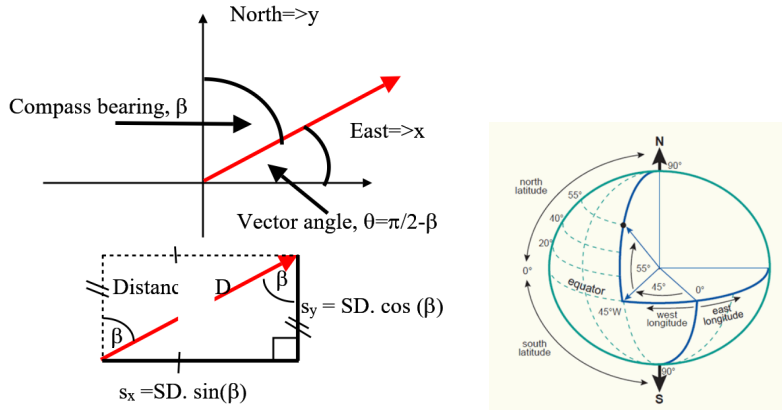[13]Ships chronometers at the time lacked sufficient accuracy, and hour glasses were often used instead.
[14]`https://www.history.com/news/why-is-a-ships-speed-measured-in-knots`

Figure 20: Co-ordinates (a) relate compass bearing to vector angle and distance travelled (b) definition of longitude and latitude `https://kartoweb.itc.nl/geometrics/Coordinate%20systems/coordsys.html`. The indicated point is longitude $\lambda = 45°$ and latitude $\phi = 55°$. Radial distance from indicated point to earth's axis is $R = R_{earth} cos(55°)$.

$$\vec{\Delta r} = SD\,sin\beta\,\vec{i} + SD\,cos\beta\,\vec{j}$$

where SD is the distance travelled daily. The change in latitude is,

$$\delta\lambda = \frac{\Delta r_y}{R_{earth}}\,radians = \frac{SD\,cos\beta}{R_{earth}} \times \frac{180°}{\pi} = \frac{SD\,cos\beta \times 180°}{3440\pi} = \frac{SD\,cos(\beta)}{60}°$$

Where on 14 August $SD = |\vec{s}| = 70N.miles$ and $\beta = 169°$, so

$$\lambda = -23.5 + \frac{70 \times cos(169°)}{60}° = -24.65°$$

In calculating the longitude, ϕ, I took the latitude on 14 August as constant and equal to $\lambda = -23.5°$. I used this to find an arc radius that was somewhat less than $R_{earth} = 3440N.miles$.

We get the final co-ordinates, on 8 October, as $-44.0°$ North and $-177.5°$ West.

3. Week 3 - Direction and Forces

We start with a summary of last week's homework, based on blogs and point out that we have expressed daily distances in terms of vectors, but also we are in effect trying to use integration.

$$\lambda = \lambda_o + \int_0^T v\frac{cos\beta}{60}dt$$

$$\phi = \phi_o + \int_o^T v\frac{sin\beta}{60cos\lambda}dt$$

where here $v = |\vec{v}|$ indicates speed or velocity magnitude. Technically the distance travelled each day (SD) is a speed. The tricky part is specifying the dependence of β, v, λ on time.

We have talked about vector addition and this is important for structures and mechanisms. For example in a crank slider (`https://www.britannica.com/technology/slider-crank-mechanism`) - typically associated with piston engines - the connecting rod points at an angle to the direction of travel. Likewise in a scotch yoke the velocity of the pin points differently to that of the yoke (https://www.youtube.com/watch?v=HhX-8RyP214). Of interest are (1) how forces can be added (2) how a force can be resolved into a particular direction (3) how force and radius might be multiplied to give torque.

When a torque is applied (e.g. a torque or socket wrench [15] the force and radial vector should be at right angles. If not we get components of torque in two or three axes. In aircraft this leads to pitch, roll and yaw.

3.1. Speaker

The speaker talks about the role of Mathematics in their engineering work.

3.2. Forces on a sail - the dot product ♣ ♣ ♣

Warning - I have simplified the application of vectors to make our lives easier.

The dot product of two vectors is,

$$\vec{a} \cdot \vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})(b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$

$$= a_xb_x + a_yb_y + a_zb_z = ab\,cos\alpha$$

where α is the angle subtended between the two vectors. We shall shortly refer to the applied "wind force". If this acts at angle α to the vessel's velocity vector the cross product $\vec{F}_W \cdot \vec{v}$ gives the power that

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[15] `https://en.wikipedia.org/wiki/Torque_wrench`

propels a vessel. Fluid dynamicists attribute two "wind forces" to two different physical effects. Drag operates in the direction of the apparent wind, observed from the vessel, and (crudely) is about air friction. Lift operates at right angles to the apparent wind, and is about the action of aerofoils.

Example: Vector addition and dot product ♣ ♣ ♣

A marine engineer intends to fit a sailing ship with auxiliary engines. In a typical test the apparent wind speed is $v_{w,app} = 30 \ m/s$. The speed of the ship is $v_{ship} = 7 \ m/s$ and the ship sails on a close reach with a boat heading angle 75° off the wind [a]. Find the total force acting on the sails, and estimate the power driving the ship.

There are two parts to the force on the sails. The drag force acts in the direction of the wind according to $F_D = 3750(kg/s)v_{w,app}$. [b] The lift force acts at right angles to the wind direction and its magnitude is six times larger than that of the drag force.

[a] https://www.nauticed.org/freesailingcourse-m1-34
[b] This is simplified - the true magnitude of force is proportional to the speed of the wind squared.

Problem Statement: Resultant wind force on a ship and consequent power.

Diagram: We align the x-direction ($\vec{i}$) with the apparent wind direction. Thus drag operates in the x-direction and lift in the y-direction.
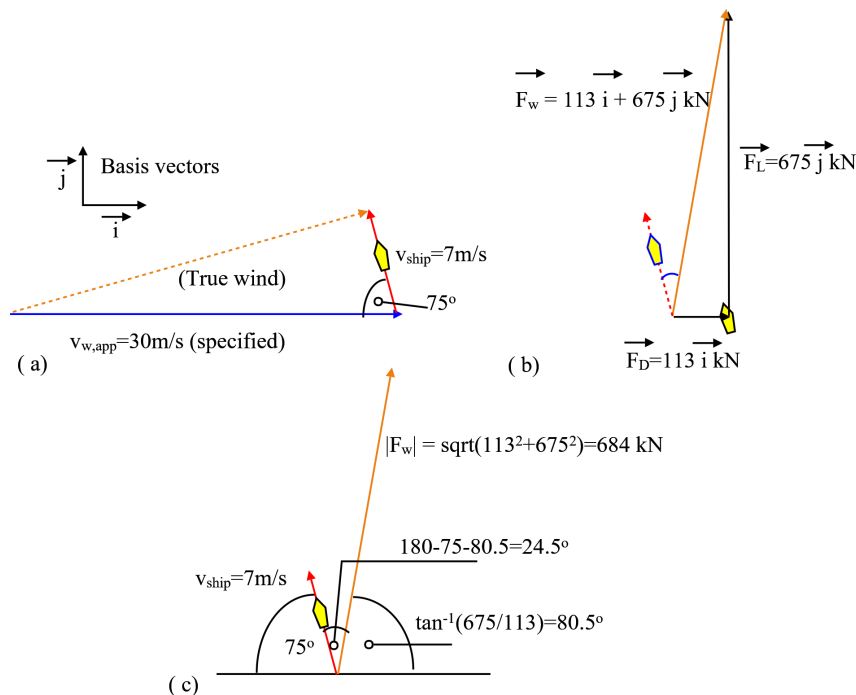

Figure 21: Force and power (a) velocities of apparent wind and ship (b) alignments (i) both wind direction and drag force with x-axis (ii) lift force with y-axis (iii) resultant total wind force F_W (c) geometric calculation of angle between ship's trajectory and wind force (24.5°)

.

Assumptions: Force is applied by wind forces only - e.g. there is no additional 'boost' from currents.

Physical principles: Work definition, if $work = force \times distance$ then $power = work/time = force \times distance/time = force \times velocity$.

Calculation: Find the drag and lift forces and sum to get total lift force

$$\vec{F}_D = 3750 \times 30\vec{i} = 112\,500\vec{i}N = 112.5\vec{i}kN$$

$$\vec{F}_L = 112.5 \times 6\vec{j} = 675\vec{j}kN$$

$$\vec{F}_W = F_D + F_L = 112.5\vec{i} + 675\vec{j}kN \quad orange\ line$$

(The wind force is coloured orange on the above diagram.) For chosen axes, the vector angle is $\theta = 180 - 75 = 105°$ and the ship velocity is,

$$\vec{v}_{ship} = 7cos(105°)\vec{i} + 7sin(105°)\vec{j} = -1.81\vec{i} + 6.76\vec{j}\ ms^{-1}$$

Now, if force and velocity were aligned it would be simple to state that $power = force \times speed$. The misalignment of force and velocity is dealt with by means of the *dot product*, otherwise known as the *scalar product*. The dot product of two vectors, $F \cdot v$ is "a scalar quantity equal to the sum of pairwise products of coordinate vectors F and v" - (see https://onlinemschool.com/math/assistance/vector/multiply/ and https://www.mathsisfun.com/algebra/vectors-dot-product.html)

$$power = \vec{F}_w \cdot \vec{v}_{ship}$$

$$power = (112.5\vec{i} + 675\vec{j}) \cdot (-1.81\vec{i} + 6.76\vec{j})kW$$

$$power = (112.5 \times -1.81)(\vec{i} \cdot \vec{i}) + (675 \times 6.76)(\vec{j} \cdot \vec{j})\ kW$$

$$power = 4359\ kW$$

(In the above $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = 1$ and $\vec{i} \cdot \vec{j} = 0$)

Discussion: An alternative version of the dot product is

$$power = |\vec{F}_w||\vec{v}_{ship}|cos\alpha = 684 \times 7 \times cos(24.5°) = 4357kW$$

where $\alpha = 24.5°$ is the angle subtended between boat heading and wind force (see diagram).

Discusion: We search the suppliers and, for example, find a $4.7MW$ diesel unit at https://www.wartsila.com/marine/build/engines-and-generating-sets/diesel-engines/wartsila-31. The power

specified in this imagined example is several times more than the 600 kW rating currently used [16].

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[16]https://www.sea.museum/whats-on/vessels/hmb-endeavour/ship-specifications

A vertical window has an area of $A = 6m^2$. At 16:00 hours , the intensity of direct solar irradiation has magnitude $I_{solar} = 250 \ W/m^2$. Standing on a patio flush with the window I take the photograph shown below. Estimate the solar energy passing through the window.

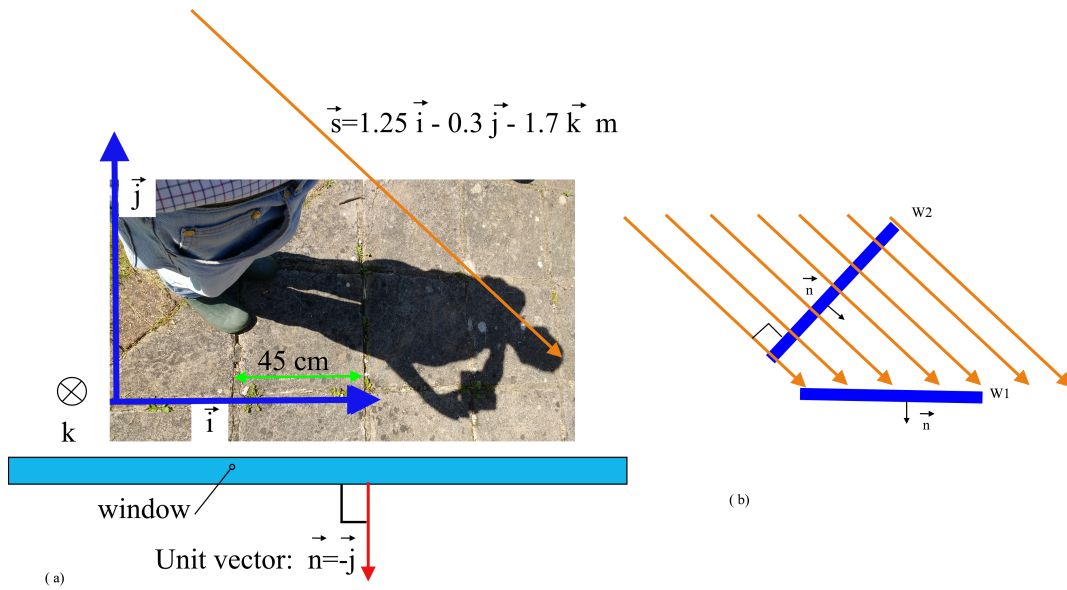Problem statement: Find solar heating effect.

Diagram:

Figure 22: Direction of solar radiation (a) indicated by my shadow. Vector $\vec{s}$ is the length from the top of my head to its shadow counterpart (b) window W1 at obtuse angle to light rays admits fewer rays than W2.

Assumptions: Ignore solar radiation absorbed by or reflected from the window. Ignore diffuse irradiation.

Calculation: The vector from the top of my head to it's shadow counterpart lies along the irradiation path. I am $1.7m$ tall and by measuring distances on my patio the displacement vector s is

$$\vec{s} = 1.25\vec{i} - 0.3\vec{j} - 1.7\vec{k}m$$

where $\vec{i}$ is aligned with the face of the window, $\vec{j}$ is horizontal and at right angles to the window, and $\vec{k}$ points vertically upwards. Vector $\vec{s}$ can be divided by its magnitude and hence normalised (length made equal to one) to give the direction of solar radiation.

$$\vec{I} = I\frac{\vec{s}}{|\vec{s}|} = 250\frac{1.25\vec{i} - 0.3\vec{j} - 1.7\vec{k}}{\sqrt{1.25^2 + 0.3^2 + 1.7^2}} = 147\vec{i} - 35\vec{j} - 200\vec{k} \ W \ m^{-2}$$

Ideally, this vector would align with the unit vector normal to the window and pointing into the room,

40

$\vec{n} = -1\vec{j}$. In fact rays strike the window at an acute angle. Energy passes through the window at a rate given by

$$E = A\vec{n} \cdot I = 6(-1\vec{j}) \cdot (147\vec{i} - 35\vec{j} - 200\vec{k}) = 6 \times (-1) \times (-35) = 210 \; W$$

3.3. Torque, roll, and the cross product ♣ ♣ ♣

This is intended to serve as useful preparation for Years 1 and 2 of study but should it prove too difficult then wait until it reappears in next year's lectures. You might well find it useful to review the vector cross product at the "Mathsisfun" website. [17].

The cross product, or vector product, allows us to calculate a new vector that is perpendicular to two other vectors, and this section talks about some applications of why we would need it. These include torque, magnetic force on a current, and corriolis force.

Figure 23: Heeling of a ship

The wind force drives a sailing vessel. It also causes it to roll or "heel", that is to "twist" normal to its axis. This brings us to the concept of torque and we shall start with the wrench in Figure 24.

In Figure 24(a) the applied force F is at right angles to the wrench and torque follows the simple formula $\tau = Fr$ where r is the effective handle length. So if the force is 150 Newtons and the wrench is 0.5 metres long then the torque is $150 \times 0.5 = 75 Newton\ metres$.

We lose something here - the direction of the torque. So instead let us write torque and ~~radius~~ position as vectors.

$$\vec{\tau} = \vec{F} \times \vec{r} = 150\vec{i} \times 0.5\vec{j} = 75\vec{k}\ N$$

The torque turns anticlockwise, rotating around the z-axis, at right angles to the force vector (x-axis) and position vector (y-axis). The $\vec{k}$ component tells us about this third axis. In particular the z-axis goes through the centre of a nut, located near N. In part (b), in a ham-fisted way, I push the wrench handle in the y-direction $(\vec{j})$. The force is now a perpendicular distance from nut N in the z-direction $(\vec{k})$ and rather than

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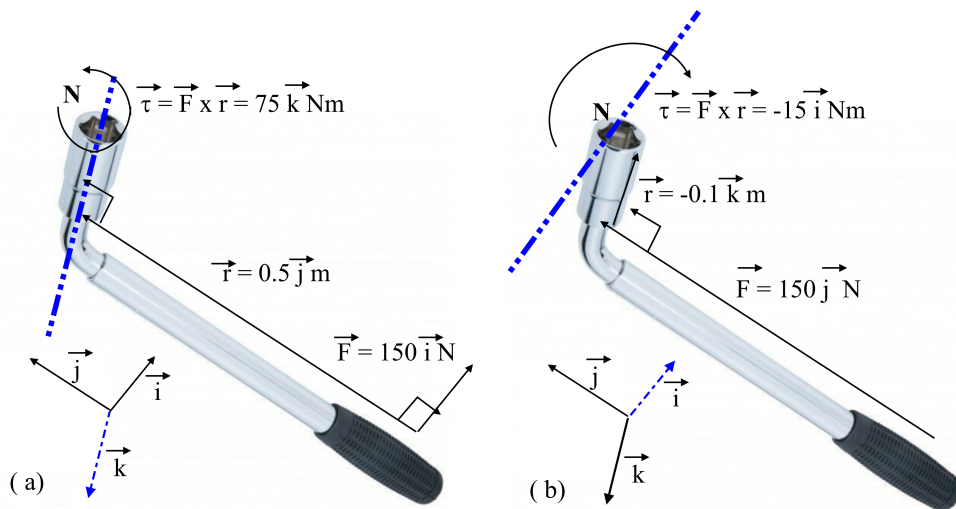[17]https://www.mathsisfun.com/algebra/vectors-cross-product.html

Figure 24: Force, position and torque illustrated with a wrench (a) good technique, pushing at right angles to handle and rotating about z-axis ($\vec{k}$) (b) bad technique, pushing along axis of handle and rotating about and x-axis ($\vec{i}$)

the handle the position vector is defined by the extension bar. There is now a misaligned torque around an axis pointing in the x-direction ($\vec{i}$) and through N. In truth there is always an inevitable small misalignment of applied force so that vectors are not strictly *orthogonal* (at right angles); we resort to cross products.

The convenient calculator at OnlineMSchool [18] alludes to the use of determinants. My own preference is the "cross product circle trick", shown graphically below.

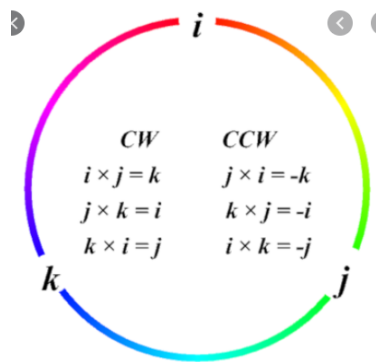Figure 25: The cross product circle trick

So in the clockwise direction.

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$

On Figure 25, reversing direction causes a sign change. The cross product of two aligned vectors is zero, so $\vec{i} \times \vec{i} = 0$ etc.

---

[18] https://onlinemschool.com/math/assistance/vector/multiply1/

Misalignment brings about torques on three axes. In machinery, the torques caused by misalignment result in wear and fatigue. If we can calculate the cross product we can try to minimise unwanted components so as to minimise the wear. Aircraft are subject to pitch, yaw, and roll [19].

In a ship, the designer must lower the center of gravity to counteract roll (healing), pitching drives either the stern or the bough into the water (hence dinghy sailors shift their weight to the front or back of the boat), and steering must be corrected to account for tendencies to yaw.

---

[19]https://en.wikipedia.org/wiki/Aircraft_principal_axes

Example: Healing of a ship ♣ ♣ ♣

A force of $684kN$ applies in a direction at $\theta = -24.5°$ to a ship's direction of travel. The force can be considered to act at a single point on a single mast, 20 metres from the axis of rotation along the ship. The ship heels by $10°$. Find the associated torque vector.

Problem statement: Torque components causing heeling (roll), pitch and yaw of a ship.

Diagram:
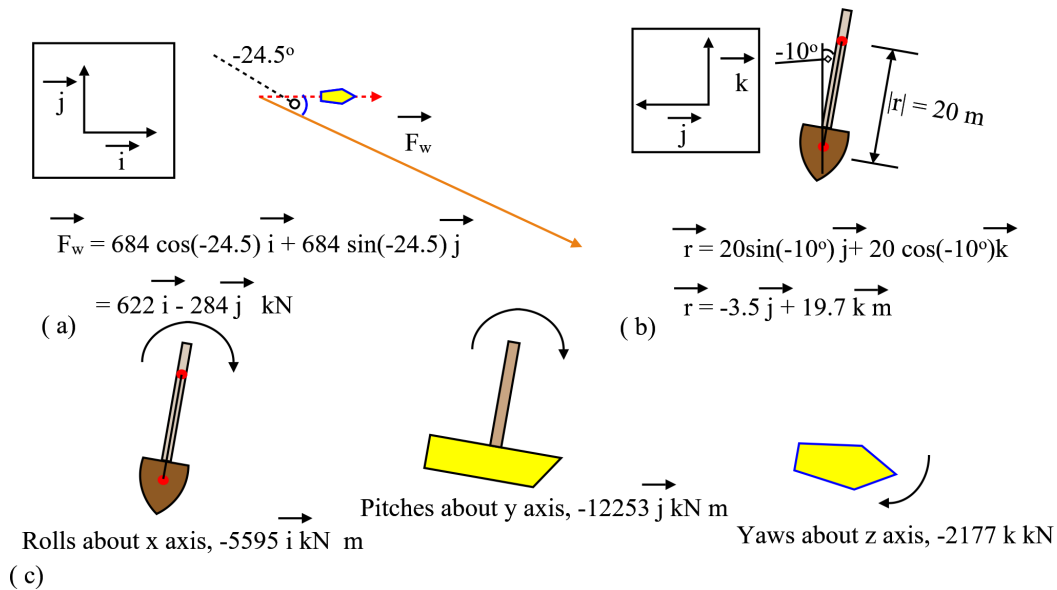
Figure 26: Torque acting on a ship, developed as a vector (a) angle between ship direction (x-direction) and applied force, shown in xy plane (b) rotation of ship and mast, shown in yz plane (c) tendency of torque to produce roll, pitch and yaw

Assumptions: Ignore any distortion of position vectors owing to pitch and yaw.

Calculation: Align the ship along the x-axis, so that it's velocity would have $\vec{i}$ component only. Resolve vectors as shown on the diagram so that

$$\vec{F}_W = 622\vec{i} - 284\vec{j} \ kN$$

$$\vec{r} = -3.5\vec{j} + 19.7\vec{k} \ m$$

The hard way. Refer to the cross product trick (Figure 25) and look for $\vec{j}$ and $\vec{k}$ to get the torque $\vec{i}$ part:

$$-284\vec{j} \times 19.7\vec{k} = -5595\vec{i}$$

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Take similar approaches to get the other two components in the complete torque vector

$$\vec{\tau} = \vec{F}_w \times \vec{r} = (622\vec{i} - 284\vec{j}) \times (-3.5\vec{j} + 19.7\vec{k}) \ kN \ m$$

$$= (-284 \times 19.7)\vec{i} + (-1 \times 622 \times 19.7)\vec{j} + (622 \times (-3.5))\vec{k}$$

$$= -5595\vec{i} - 12253\vec{j} - 2177\vec{k} \ kNm$$

The easy way: Use the calculator at OnlineMSchool [20].
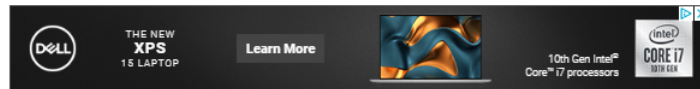
Figure 27: Using the cross product calculator

Discussion: As sketched on the diagram, the three torque components will induce roll ("healing"), pitch and yaw.

Other examples of cross products are

- Coriolis force, $\vec{F}_{cor} = -2m\vec{\Omega} \times \vec{v'}$

- Force on an electrical charge in a magnetic field, $\vec{F}_m = \vec{B} \times \vec{I}L$ or $F_m = \vec{B} \times \vec{v}q$

3.4. Recap

We shall use this opportunity to catch up with tricky aspects that arise on blogs etc, and to recap these. These items have been covered so far.

- Layout of calculations: problem statement, diagram, assumptions, physical laws, calculation, description;

- trigonometry, exponential decay, spreadsheet use;

- integratal = area under curve;

- differentiation = curve slope ;

- separate variables and integrate to solve differential equations;

- a vector and its basis vectors, $\vec{v} = v_x\vec{i} + v_y\vec{j}$

- vector addition, align vectors head to tail;

- vector dot product, $Power = \vec{F} \cdot \vec{v}$, when vectors should be aligned;

- - vector cross products, $\vec{\tau} = \vec{F} \times \vec{r}$, when vectors should be at right angles;

In the next two sessions we shall cover

- - stationery points, $dy/dx = 0$ might mean least cost;

- - partial differentials, two or more independent variables. To differentiate w.r.t. x_1 hold $x_2, x_3...$ constant

- recording and maintaining data;

- presenting data.

> **Homework: Drift velocities ♣ ♣**
>
> (A) Complete the spreadsheet activity from last week. (B) Generally the ocean currents can be ignored. However, from 9 September onward the observed velocity of driftwood is 0.61 knots, on a bearing of 325°. Use vector addition to information to recalculate the course estimated in week 2. Complete column G in the third sheet of MathsHomework.xls. (C) If time allows, experiment with online vector calculators.

Tips and hints

(A) See hints given for the last session.

(B) The displacement of driftwood in one day is

$$\vec{\Delta r}_{drift} = 24 \times 0.61 \times (sin(325)\vec{i} + cos(325)\vec{j})$$

Add this to the ship's displacement relative to the ocean (measured with magnetic compass and Dutchman's log). The relative velocity is what you estimated last week.

$$\vec{\Delta r} = \vec{\Delta r}_{drift} + \vec{\Delta r}_{rel}$$

(On the LHS, I omit subscripts indicating absolute or "true" displacement). Over 24 hours you should find that the change in latitude is something like.

$$
\begin{aligned}
\delta\lambda &= \frac{\Delta r_y}{R_{earth}} \; radians \\
&= \frac{SDcos\beta + 12}{R_{earth}} \times \frac{180\,°}{2\pi} \\
&= 0.2 + \frac{SDcos(\beta)\,°}{60}
\end{aligned}
$$

This has been put in the Session 3 worksheet of mathsHomework.xls. Formulate an expression for change in longitude and put it in the spreadsheet.

Remember, when using trigonometric functions, to convert degrees to radians.

(C) Try the vector plotter at `https://academo.org/demos/3d-vector-plotter/`. For their default values, try out vector addition and cross product. Then set up vectors that lie along the x and y axes and try vector addition and cross product.

Find the vector tools at `https://onlinemschool.com/math/assistance/vector/p_to_vector/`.

4. Week 4 - Best Solutions

We start with a summary of homework responses.

We often seek minimum cost of materials or maximum efficiency. This is about finding and investigating stationary points.

4.1. Speaker

The speaker talks about the role of Mathematics in their engineering work.

4.2. Least cost ♣ ♣

Let the economic cost of a system, machine or building be y and let this depend on two related effects, both dependent on a variable x. For example,

$$y = ax + \frac{b}{x}$$

At a *stationary point $dy/dx = 0$* and $y(x)$ is a minimum, a maximum or a point of inflection.

$$\frac{dy}{dx} = a - \frac{b}{x^2}$$
$$\frac{d^2y}{dx^2} = +2\frac{b}{x^3}$$

At the stationary point,

$$\frac{dy}{dx} = 0$$
$$x = \sqrt{\frac{b}{a}}$$
$$\frac{d^2y}{dx^2} = +2a\sqrt{\frac{a}{b}}$$

Provided a, b are positive the solution will be a minimum, because $\frac{d^2y}{dx^2} > 0$. The solution is shown for two particular values of a and b. The minimum (square box) is evident and one notes that gradient is positive to the right of the stationary point and negative to the right, confirming $\frac{d^2y}{dx^2} > 0$.

So long as coefficients a and b are positive, then the second derivative is positive and we have a local minimum. This value of x might, for instance, offer the cheapest option. Figure 28 shows the trend. Note how the function tends to $a.x$ near the left of the picture and b/x near the right.
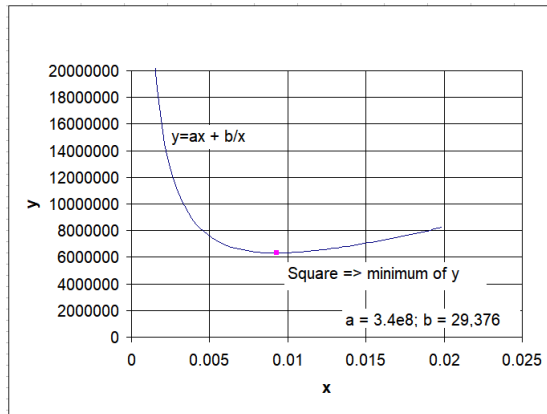
Figure 28: Function $y = ax + b/x$ including minimum value

Example: Power losses through copper wire ♣ ♣

In a simple electrical microgrid current is transmitted via copper wires. The load is 120 amps throughout the circuit, the total length of wire is 4000 metres, and the operating lifetime of the wires is 100, 000 hours. The cost of copper is \$85,000 per cubic metre (to include working and manufacture) and the electrical resistivity is $17 \times 10^{-9} Ohm/m$. Electricity is sold for \$0.30 per kilowatt hour. Find the optimum amount of copper.

Problem statement: Wire diameter producing minimum total cost (capital + operating losses) of electrical transmission wire.

Diagram: See Figure 28. (I come clean here - it comes from an earlier piece of work with the wrong value of b for this particular piece of work. It can be edited in the online file mathsSummerSchoolExamples.xls.)

Assumptions: No copper is lost in manufacturing processes, the cost of drawing the wire (through dyes) and braiding it is independent of wire diameter. Costs of electrical insulation, pylons, and labour are ignored (or assumed independent of the mass of copper used).

Physical Laws: Ohms law - gives us power loss $= I^2 R = I^2 L\rho/A$ where ρ is resistivity, L is length, A is cross sectional area.

Calculations: There are two costs - the capital cost of the copper and the operating loss from resistance heating. Let A be the cross sectional area of the wire; the cost of wire is then a multiple of wire volume

$$C_{cap} = \$85,000 \times L \times A = \$340,000,000A$$

The loss of sales revenue is treated in the same way as an operating cost. Over ten years it becomes

$$C_{op} = 100,000 \; hours \times I^2 \; R \times [1kW/1000W] \times \$0.30/(kWhr)$$

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$$C_{op} = 30\$/W \times 120^2 \times \frac{L\rho}{A}$$

where ρ is the electrical resistivity of copper. Put in $L = 4000m$ and $\rho = 17 \times 10^{-9} Ohm/m$ to the above

$$C_{tot} = C_{cap} + C_{op} = \$(340,000,000A + 29.4/A)$$

Referring to the previous work, substitute C_{tot} for y, A for x and appropriate numbers for a, b ; the best value of cross sectional area is

$$A_{opt} = \sqrt{\frac{29.4}{340,000,000}} = 2.9 \times 10^{-4} \ m^2$$

And this corresponds to a volume of $1.2m^3$ (about 10 metric tonnes).

Discussion: The corresponding cost is $102\,000$ and the wire diameter is 1.5 cm. The mass of the cables might require strong pylons, so in truth the optimisation would have to be reformulated to account for the cost of pylons. The designers might consider transmitting with a higher voltage but lower current.

4.3. Two independent variables ♣ ♣ ♣

Often a performance parameter is influenced by two or more independent variables. So the parameter can change with respect to variable 1 *or* variable 2. This introduces us to the topic of partial derivatives. See `https://www.mathsisfun.com/calculus/derivatives-partial.html` .

A simple illustration is the hiker's contour map. A contour is a curve showing a constant height; when I walk along a contour I neither gain height nor lose it. The two independent variables are east-direction(x) and north-direction(y) whereas the dependent variable is height above sea level (z). At the indicated origin, a movement to the north crosses few contours and comparatively little height is gained or lost (you might notice the nearby track on the photograph). Movement to the east crosses contours quickly and height is lost (towards the small lake at the base of the cliff). The contour plot is a well known way of plotting one dependent variable against two independent variables.

(a)
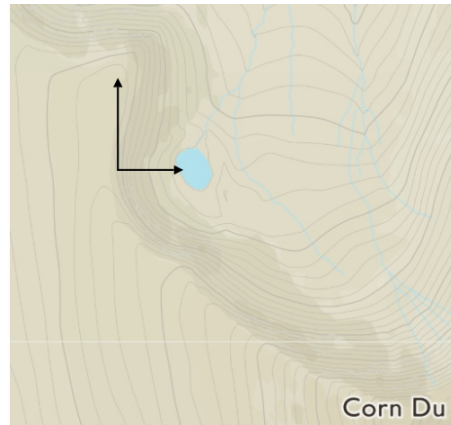
(b)

Figure 29: Welsh countryside (a) photograph, with arrows pointing north and east (b) corresponding contour map - height is constant along a contour. See `https://osmaps.ordnancesurvey.co.uk/51.88627,-3.44434,15`

In the fuel consumption map below the parameter of interest is the "specific fuel consumption", SFC, the mass of fuel required to develop one unit work (an inefficient engine has a high SFC). The contours (squashed circles) show constant SFC. The x-axis represents engine speed (in revolutions per minute) and the left hand, y1-axis represents "brake mean effective pressure" - the average pressure in the cylinder (found by integration). (The power is $power(kW) = 2.5 \times 10^{-3} \times BMEP \times RPM$. You will come across engines and engine performance parameters in the laboratory in the next two years. This particular figure comes from measurements rather than an equation.)

The lowest fuel consumption is marked by the red cross, X. This is at about 2200 RPM (x-axis) and a BMEP of 8 bar.

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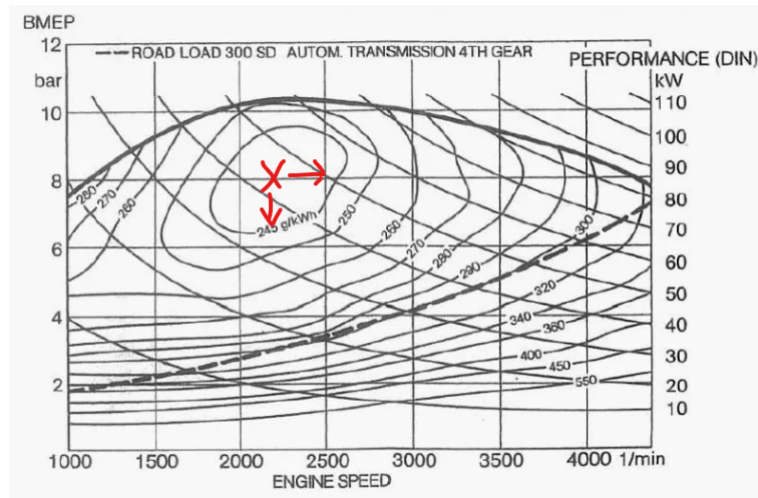
Figure 30: Fuel consumption map - an example of two independent variables. See `https://www.eng-tips.com/viewthread.cfm?qid=363722`

Example: Optimum Engine Speed ♣ ♣

The BMEP is 5.5 bar. At what engine speed would we find optimum SFC?

Find 5.5 bar on the Y1 axis. Reading horizontally the minimum SFC is approximately 250 g/kWh at approximately 1750 RPM. This minimum is rather fuzzy.

Example: Optimum power ♣ ♣

The engine is to be operated at 3500 RPM. For what BMEP would the SFC be an optimum?

The best SFC lies between 270 and 280 g/kWh, by eye I get 276 g/kWh at 8 bar. This is a clearer minimum.

To illustrate how partial differentiation works with analytical expressions, let us take engine speed as x, BMEP as y, and SFC as z. For purposes of illustration, consider a very simple function that might give SFC (this is not the same as Figure 30).

$$z = 235 \times (1 + (x - 2300)^2 / 4 \times 10^6) \times (1 + (y - 8)^2 / 100)$$

For a fixed BMEP (y), changes in engine speed (x) can either increase or reduce SFC (z). (For instance, if the road incline were to become steeper constant BMEP is kept by reducing engine speed). One might write

$$\frac{\partial z}{\partial x} = \frac{235}{4 \times 10^6} (2(x - 2300))(1 + (y - 8)^2 / 100)$$

In terms of *mathematics*, notice that y(BMEP) has been fixed and treated in the same way as any other

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constant. The only parameter subject to differentiation laws is x(RPM) because we have not allowed y to change. In terms of *engine operation*, we find that changing engine speed might either increase or decrease the SFC(z) - that is the mass of fuel needed to produce one unit of work.

Now consider a change in the y-variable(BMEP). This time x (RMP) is treated as a constant whereas y (BMEP) is operated on.

$$\frac{\partial z}{\partial y} = 235 \times (1 + (x - 2300)^2/4 \times 10^6) \times (2(y - 8)/100)$$

Wolfram Alpha's "partial derivative calculator" allows simple experiments. Allow plenty of time for the window to refresh. It is probably best found with a search engine but otherwise try `https://www.wolframalpha.com/widgets/gallery/view.jsp?id=35764b7940c52f2ba9a2f2586c094ee6`

4.4. Best of two ♣ ♣ ♣

Sometimes we want to know the best conditions subject to two or more independent variables. We seek values of variables x, y where

$$\partial z/\partial x = 0 \qquad \partial z/\partial y = 0$$

> **Example: Rocket science ♣ ♣ ♣**
>
> A maroon summons assistance in two modes of operation. (1) At time $t = \tau$ all propellant fuel has been used and a visible flare is ignited instantaneously. (2) Also at $t = \tau$, the maroon drops an SOS message by parachute, gliding on the original bearing, in a straight trajectory inclined at 30° to the horizontal.
>
> There are two adjustable variables - the angle at which the maroon thrust is inclined to the horizontal, θ, and the burn time, τ. Note that maroon thrust is in inverse proportion to τ. If the maroon were fired in a horizontal direction its acceleration, unaffected by gravity, would be $a = (200/\tau) \ m/s^2$.

Problem Statement: Find burn time and elevation settings that give (1) maximum height of a maroon after fuel is exhausted (2) maximum travel of a parachute message. Ascent ends when fuel is exhausted.

Diagram: See Figure 31

Assumptions: We ignore drag forces and assume that a constant rate of burn gives a constant acceleration. We assume that the propellant mass is far smaller than total rocket mass, so that loss of propellant has negligible impact on acceleration.

Principles: We can use SUVAT in the form $S = 0.5aT^2$ (zero initial velocity or travel) or integration. We need partial differentials. Distance, velocity and acceleration are written using basis vectors.
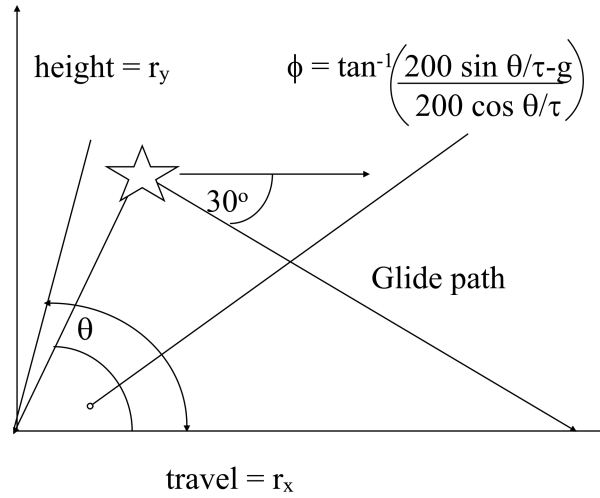
Figure 31: Maroon Trajectory

Calculation: The first part of the calculation concerns the use of vectors and SUVAT. The second part concerns partial differentiation.

The vectors and SUVAT part

Allowing for elevation and gravity, one finds that acceleration of the rocket is,

$$\vec{a} = +\frac{200cos(\theta)}{\tau}\vec{i} + \left(\frac{200sin(\theta)}{\tau} - g\right)\vec{j} \qquad at\ t < \tau_1$$

(In the above, we have to subtract the downwards acceleration due to gravity: for example, if the rocket were directed vertically $\theta = \pi/2$ and net force would be $F_v = m(200/T - g)$). The conditions at time $t = 0$, initial conditions, are displacement and velocity

$$\vec{r_0} = 0\vec{i} + 0\vec{j} \qquad at\ t = 0$$
$$\vec{v_0} = 0\vec{i} + 0\vec{j} \qquad at\ t = 0$$

In general form, SUVAT gives $S = aT^2/2$ (there is no initial velocity) and this can be applied separately to the two components of the acceleration vector. Then at the end of the burn time τ the position vector is,

$$\vec{r}(\tau) = 100cos(\theta)\tau\vec{i} + (100sin(\theta)\tau - 0.5g\tau^2)\vec{j}$$

We can identify horizonal displacement as r_x, and vertical displacement as r_y. This is at the end of the

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burn time. Note that the ultimate travel includes the glide path - a multiple of height.

$$height = r_y = 100sin(\theta)\tau - 0.5g\tau^2$$

$$travel = r_x + \frac{height}{tan(30°)} = 100cos(\theta)\tau + \sqrt{3}(100sin(\theta)\tau - 0.5g\tau^2)$$

The partial differentiation part

The conditions for maximum height are

$$\partial height/\partial\tau = \partial h/\partial\theta = 0$$

The rocket should be directed vertically because,

$$\partial height/\partial\theta = 0 \implies 100cos(\theta_m)\tau = 0 \implies \theta_m = \underline{\pi/2 = 90°}$$

And the burn process should be adjusted to give the best burn time.

$$\partial height/\partial\tau = 0 \implies 100sin(\theta_m) - g\tau_m = 0 \implies \tau_m = 100sin(\theta_m)/g = 100/g = \underline{10.2s}$$

You will note in the above that we use subscript $_m$ to indicate optimum conditions. On the first line we leave τ unchanged and operate with θ. On the second line we leave θ unchanged on operate with τ.

Now let us turn our attention to optimum travel.

$$\partial travel/\partial\theta = 0 \implies 100\tau(-sin(\theta_m) + \sqrt{3}cos(\theta_m)) = 0 \implies tan(\theta) = \sqrt{3} \implies \theta_m = \underline{\pi/3 = 60°} \quad (1)$$

$$\partial travel/\partial\tau = 0 \implies 100cos(60°) + 100\sqrt{3}sin(60°)) - \sqrt{3}g\tau_m = 0 \implies \tau_m = \underline{11.8s} \quad (2)$$

Discussion: Some checks are carried out in the file mathsSummerSchoolExamples.xls. If you set $\theta = pi/2$ you will find the maximum height of 510 m close to $\tau = 10s$, close to our optimum. If you set $\theta = pi/3$ then the optimum travel of 1177 m is found close to $\tau = 12s$.

Perhaps more importantly, the engineer should question the benefits of gaining 300 metres of glide path at the cost of 180 metres of altitude. See `https://rechneronline.de/sehwinkel/distance-horizon.php`. When fuel is exhausted, the maroon still has upwards momentum, so would have achieved greater height had ignition of the flare been delayed.

4.5. Homework ♣ ♣ + ♣ ♣ ♣ ♣

Complete last session's calculation of ship's course. For the exercise below attempt statements of problem statement, assumptions, diagram, physical laws and discussion (♣ ♣). The calculation is difficult and optional (♣ ♣ ♣ ♣) - we provide only one tip but we also provide a worked solution.

<div style="border:1px solid; padding:10px;">

Example: Fuel and cargo weight ♣ ♣ ♣ ♣

Suppose our vessel is to be fitted with auxiliary engines, so that if it were to be demasted it could "limp home" to shore. The total journey is 4000 km but in the worst case the vessel need only have a range of 2000 km. The ship can be sure of being spotted and rescued if it travels in a 2000km-long straight line and along a bearing from north of $\theta = \pm 12°$

When resolved into the ship's direction, the drag force is

$$F_D = \frac{200(v+2)^2}{cos(\pi/18 - \theta)}$$

where F_D is in kN and v is the ship's absolute speed in m/s. If $L = 2,000,000\,metres$ is the distance of travel then the total mass of fuel (in kg) is

$$m_{fuel} = \frac{F_d L}{225000}$$

and the mass of emergency supplies (in kg) is

$$m_{sup} = \frac{0.003L}{v}$$

Find the speed and bearing that minimises the total mass.

</div>

Hint and tip: Near the end of this question you will need to solve an equation looking like $v^3 + av^2 = b$. It is far easier to use a tool such as `https://www.wolframalpha.com`. E.g. try

    solve x^2−5∗x+6=0

For some solutions you will need to press the "approximate forms" tab.

SOLUTION

Problem statement: Choose boat velocity (speed and bearing) to yield least mass of fuel and supplies.

Assumptions: Students should provide these.

Diagrams: Students should provide these.

Physical Principles The work done is the product of force and distance (work definition).

Calculation: Minimise

$$m = 1778 \frac{(v+2)^2}{cos(\pi/18 - \theta)} + \frac{6000}{v}$$

with respect to v and θ.

If $\partial m / \partial \theta = 0$ then

$$1778(v+2)^2 \frac{sin(\pi/18 - \theta)}{cos^2(\pi/18 - \theta)} = 0$$

The optimum course is $\underline{\theta_m = \pi/18 = 10°}$.

If $\partial m / \partial v = 0$ then

$$\frac{\partial m}{\partial v} = 3556 \frac{v+2}{cos(\pi/18 - \theta)} - \frac{6000}{v^2}$$

$$v^3 + 2v^2 - 1.69 cos(\pi/18 - \theta) = 0$$

Given the optimum value of θ

$$v^3 + 2v^2 - 1.69 = 0$$

A useful tool is at `https://www.wolframalpha.com/calculators/equation-solver-calculator/`. Type

```
solve v^3 + 2 v^2 - 1.69 = 0
```

and click the "approximate forms" tab.

Optimum velocity is $\underline{v_m = 0.78m/s}$. The corresponding mass is 21,433kg - about 13.7 metric tonnes of fuel and 7.7 metric tonnes of supplies.

5. Week Five: Handling Data

For this section it is better to follow the online videos and discussion therein. Some notes are attached here for completeness.

5.1. Speaker

Dr Selwyn will introduce this section.

5.2. Importance of data

Data matters. It is at the core of every decision made. Vast amounts being collected and generated all the time. For engineers, it helps us to answer questions and find areas for improvement (e.g. Which material is strongest? Where is the leak in the pipe? Which turbine has stopped working? Which fault is the most common and how can we stop it happening?)

If data is not recorded properly, then it becomes useless. For example the in ship's log book shown in earlier weeks - there is no column for longitude. Tables of results without headings, or without units are ineffective.

Labels for tables: Make sure each column of a table has a header row with what is in it and its units (do not mix numbers and units in Excel cells). Use consistent significant figures in each column. Often, however, there is too much data to easily read in table. Also, the reader cannot easily see trends in tables, and a figure (graph) is far friendlier. We need to carefully choose the best type of figure for the data we are communicating. A sketch of the expected graph is a useful addition to your log book.

In the video we give some examples of how to use a log book. In paper form, the log book keeps everything in one place. You can record data (=numbers) but also information such as the outcome of conversations with tutors. Think carefully about titles and dates, section headings, tables and figures (see above). You cannot add comments in a distinctive colour. You might want to consider electronic versions (although perhaps taking your expensive i-pad into a laboratory environment carries some risk) or "backing up" your paper version by taking photographs. Log books are important in laboratories, 3rd/4th year projects, and even for problems sheets (you might already have similar practice from previous studies - e.g. allocating a folder for each subject).

5.3. Presenting data

So now your log book is full of data, how best should you use it to convince someone of something? We need to present it clearly so it answers whatever question was being asked.

Different types of data need different graph types. For example, discrete data [21] are often plotted as a bar chart (discrete - usually data that is counted e.g. number of parts failing per day) whereas continuous

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[21]https://www.mathsisfun.com/data/data-discrete-continuous.html

Turbojet Lab

23/3/2020

Aim: investigate performance of turbojet

Notes: vary throttle from 20% to 100% of max value + record all data shown on equipment. See lab handout (on BB) for diagram

(annotations) Different units — Gauge not absolute — Convert to Kelvin)

Throttle (% max)	Speed (rpm)	Air Flow (l/s)	Fuel Flow (l/hr)	Thrust (N)	Pressure 1 (bar)	Temperature (°C) 1	2	3
20	51378	26	8.1	16	0.17	25	820	526
40	69785	49	11.1	26	0.32	46	859	535
60								
80								

Figure 32: Example of a log book page, featuring page title, date, aim, notes, column headings (with units) and annotations

data are often plotted on line graphs (e.g. temperature of a surface over time). But note that the lines connecting points represent linear interpolation between the two points: the mathematical function $y = f(x)$ corresponding to measurement points $(x_1, y_1), (x_2, y_2)$... might well be non-linear. Therefore, in engineering and science continuous data are often plotted as points (of measurement) on a graph, then a line or curve of best fit fitted to results (e.g. (1) in the first video on the area of activated carbon functions were manipulated to give a straight line form (2) material properties from a stress strain curve can be fitted to a straight line and extrapolated to find untested values). With two variables, then scatter plots are useful (e.g. ship's position is given as x and y coordinate and each point is on a different day).

Here are some key pointers for presenting graphs (we shall use the acronym **SALLT**) Use:

- **S**cales that allow trends to be evident;

- Well labelled **A**xes, explaining what is shown and including units;

- A **L**egend stating what each data series is (e.g. 'engine 1', 'engine 2');

- A **T**itle, *positioned below the Figure*, in the form 'Figure 1: Short description of figure'.

Datasets should use different markers and sensible colours (**not** yellow). Colour is useful but bear in mind that some people are colour blind and some printers work in black and white. So you use different symbols for data from engine one (*) and engine two (+).

Tables are useful for communicating precise data. Note that the table title ("Table 1 My data') should be informative and by convention should be located *above* the table.

f(x) = − 8.01989670868347E-07·x² + 0.003671066614146 x + 0.050563713235293
R² = 0.909454775024295

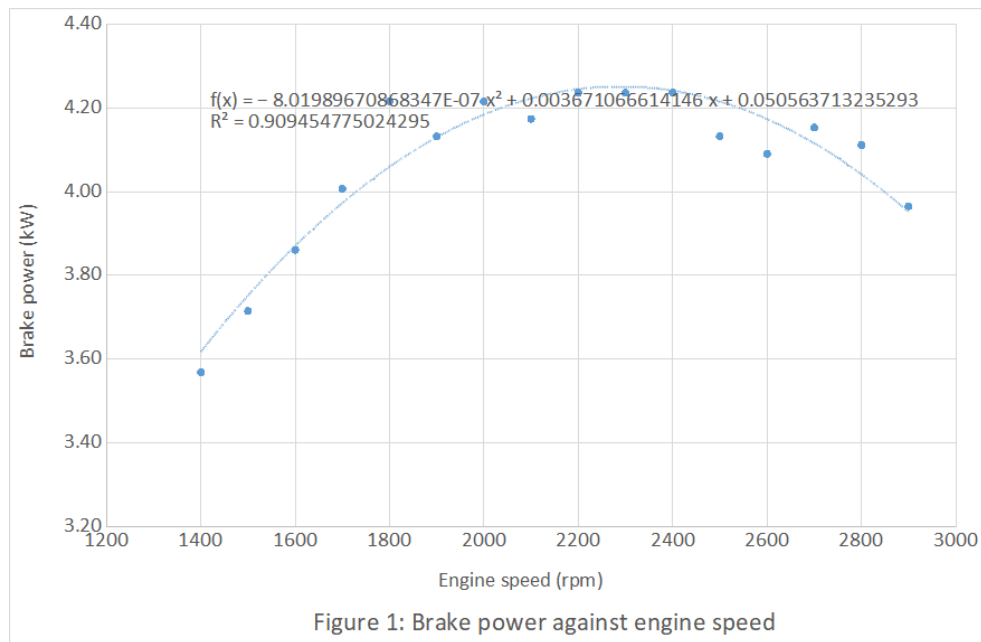Figure 1: Brake power against engine speed

Figure 33: Example of a Figure, featuring figure number and title below the Figure, axes with description and units, quadratic curve fitted to measurements ($f(x)$ and $R(x)$ were presented by Excel and show excessive (false) precision)

Table 1: Petrol engine data collected at various engine speeds.

Engine speed (rpm)	Torque (Nm)	Air flow (l/min)	Fuel time (s)	Fuel mass flow rate (kg/s)	Power (kW)	Brake thermal efficiency (%)
1400	17.1	220	67.3	0.000203	3.57	41
1500	17.8	234	59.8	0.000228	3.71	37
1600	18.5	248	51.9	0.000263	3.86	34
1700	19.2	256	47.2	0.000289	4.01	32
1800	20.2	267	41.2	0.000331	4.22	29
1900	19.8	278	39.5	0.000346	4.13	28
2000	20.2	286	37.4	0.000365	4.22	27
2100	20	297	34.8	0.000392	4.17	25
2200	20.3	309	32.4	0.000421	4.24	23
2300	20.3	321	30.6	0.000446	4.24	22
2400	20.3	335	29.5	0.000463	4.24	21
2500	19.8	342	26.5	0.000515	4.13	18
2600	19.6	354	25.5	0.000535	4.09	18
2700	19.9	365	24.5	0.000557	4.15	17
2800	19.7	376	24.1	0.000566	4.11	17
2900	19	382	23.6	0.000578	3.97	16

Figure 34: Example of a Table, featuring table number and title above the table, column headings with units, colour coding to distinguish raw data and calculated values

5.4. Analysing data

We often need to do some work on the data and results to extract the answer to the questions. We can calculate some basic statistics - for example we could apply mean, mode, median to the ship's speed. Or we might group data (e.g. 0-2 knots, 2.1-4 knots, etc).

For continuous data - our data handling depends on what we are looking for. We might want to calculate the mean etc as above.

More often might be looking for a trend, so want to fit a trend line (we can do this manually - the proof by derivation needs partial derivatives, Khan academy has good tutorials at `https://www.khanacademy.org/math/statistics-probability/describing-relationships-quantitative-data/more-on-regression/v/squared-error-of-regression-line`). Excel can do this for you for some trends, other packages e.g.

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Matlab can do more. Usually goodness of fit is described using R^2 value (tells us how much of the variation in the y value is explained by the best fit curve/line) - the higher the better. Once we have line/curve of best fit, can extrapolate/interpolate/analyse to find something out - we e.g. materials properties needing 0.1 percent strain.

Here are some links to useful Excel features

- basic statistics `https://www.excel-easy.com/functions/statistical-functions.html`

- conditional statements `https://www.exceltrick.com/formulas_macros/excel-if-statement/?=count`

- number formats (in particular, n decimal places) `https://exceljet.net/custom-number-formats`

- conditional formatting `https://www.excel-easy.com/data-analysis/conditional-formatting.html`

- adding trend lines `https://www.excel-easy.com/examples/trendline.html`

5.5. Homework

Plot the path of the ship - corrected (Week 2) vs uncorrected (Week 3, allowing for ocean current) vs measured - note discrepancies. Where does ship end up if currents not considered? Do some basic statistics on data.